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CONFORMAL GEODESIC TRANSFORMATIONS

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ABSTRACT. We treat conformal geodesic transformations with respect to submanifolds in a Riemannian manifold M. Non-isometric ones only exist when P is a hypersurface or reduces to a point. For these two cases, we derive necessary and sufficient conditions for the existence in terms of the Jacobi operator and show how this existence influences the geometry of the hypersurface and that of the ambient space. As an illustration, we treat the case of warped product spaces.

1. INTRODUCTION

Geodesic symmetries or reflections with respect to points have been generalized to *geodesic transformations* in [15]. See also [6]. The main purpose was to study such *conformal* transformations. In [7], we extended this notion to *geodesic transformations with respect to submanifolds* P of Riemannian manifolds. Roughly speaking, these transformations are local diffeomorphisms which transform a tubular hypersurface about P into another tubular hypersurface by moving points along normal geodesics of P and leaving the points of P fixed. Reflections with respect to P are natural examples. Special conformal or partially conformal geodesic transformations with respect to points and hypersurfaces are used in [7] to characterize real, complex and quaternionic space forms. In [8], we used divergence-preserving geodesic transformations to derive a new characterization of harmonic spaces. Furthermore, divergence-preserving geodesic transformations with respect to submanifolds are treated in [9]. In particular, this kind of geodesic transformations is used to characterize

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isoparametric hypersurfaces in real space forms and Hopf hypersurfaces with constant principal curvatures in complex space forms.

In this paper, we focus on *conformal* geodesic transformations with respect to submanifolds. In Section 2, we collect some material about the study of the geometry of tubular neighborhoods of a submanifold, needed to give an analytic treatment and study of these transformations. In Section 3, we derive the first results. The main property states that non-isometric conformal geodesic transformations only occur when the submanifold is a hypersurface or reduces to a point. Moreover, we show that the hypersurfaces are necessarily totally umbilical and have constant mean curvature.

Section 4 is devoted to the study of geodesic transformations with respect to points. First, we derive necessary and sufficient conditions for conformality. Then we express these conditions in terms of the Jacobi operator and its covariant derivatives and draw some conclusions from them. In Section 5, we make a similar study for conformal geodesic transformations with respect to hypersurfaces, in particular when the ambient space is locally symmetric.

Finally, in Section 6, we illustrate our technique and results for hypersurfaces when the ambient space is a warped product space and the hypersurfaces are the fibers of this product.

2. DEFINITION AND PRELIMINARIES

Let (M, g) be an n-dimensional, connected Riemannian manifold and ∇ its Levi Civita connection. R denotes the associated Riemannian curvature tensor taken with the sign convention $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for all smooth vector fields X, Y. Moreover, we put $R_{XYZW} = R(X, Y, Z, W) = g(R_{XY}Z, W)$. Furthermore, let P be a q-dimensional topologically embedded submanifold. In the rest of the paper, and for simplicity, we assume that all data are analytic although at several places smoothness is sufficient, but this will be clear from the context.

Next, let $\exp_\nu$ denote the exponential map of the normal bundle ν of P.

Definition 2.1. A *geodesic transformation* φ_P with respect to P is a map defined by

$$\varphi_P : p = \exp_\nu(ru) \mapsto \varphi_P(p) = \exp_\nu(s(r)u)$$

which leaves P invariant. Here u is an arbitrary unit normal vector of P and r and s are supposed to be sufficiently small such that φ_P is a local diffeomorphism. Moreover, the function $s : r \mapsto s(r)$ with $s(0) = 0$ is supposed to be analytic in a neighborhood of $r = 0$. For $s(r) = -r$ we obtain a (*local*) *reflection* with respect to P.

To describe analytically a geodesic transformation φ_P, we shall use Fermi coordinates. For that reason, we recall the definition as well as several aspects about the geometry of a tubular neighborhood about P. See [11], [12], [17] for more detailed information. Let $m \in P$ and let $\{E_1,\ldots,E_n\}$ be a local orthonormal frame field of (M, g) defined along P in a neighborhood of m. We specialize this field such that $E_1,\ldots,E_q$ are tangent to P. For a system of coordinates $(y^1,\ldots,y^q)$ of P in a neighborhood of m such that $\frac{\partial}{\partial y^i}(m) = E_i(m)$, $i = 1,\ldots,q$, we define the Fermi coordinates $(x^1,\ldots,x^n)$ with respect to m, $(y^1,\ldots,y^q)$ and $\{E_{q+1},\ldots,E_n\}$ by

$$\begin{cases} x^i\left(\exp_\nu\left(\sum_{q+1}^{n} t^\alpha E_\alpha\right)\right) = y^i, & i = 1,\ldots,q, \\ x^a\left(\exp_\nu\left(\sum_{q+1}^{n} t^\alpha E_\alpha\right)\right) = t^a, & a = q+1,\ldots,n \end{cases}$$

in a neighborhood of the zero section of P in ν, taken sufficiently small such that $\exp_\nu$ is a diffeomorphism. Note that these Fermi coordinates are normal coordinates when P is a point.

Furthermore, put $s(r) = \rho(r)r$ where r denotes the normal distance function to P. Here, we have $r^2 = \sum_{a=q+1}^{n} (x^a)^2$. Then φ_P is defined by

$$\varphi_P : (x^1,\ldots,x^q,x^{q+1},\ldots,x^n) \mapsto (x^1,\ldots,x^q,\rho(r)x^{q+1},\ldots,\rho(r)x^n)$$

and we have

$$\text{(2.1)}\qquad \begin{cases} \varphi_{P*}\dfrac{\partial}{\partial x^i}=\dfrac{\partial}{\partial x^i}, & i=1,\dots,q,\\ \varphi_{P*}\dfrac{\partial}{\partial x^a}=\rho\dfrac{\partial}{\partial x^a}+\rho'\dfrac{\partial r}{\partial x^a}\displaystyle\sum_{b=q+1}^{n}x^b\dfrac{\partial}{\partial x^b} \end{cases}$$

where $\sum_{b=q+1}^{n} x^b \frac{\partial}{\partial x^b} = r\frac{\partial}{\partial r}$.

In what follows, we need the expressions for the components

$$g_{ij}=g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right),\quad g_{ia}=g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^a}\right),\quad g_{ab}=g\left(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b}\right),$$

$i,\ j=1,\dots,q;\ a,\ b=1,\dots,n$ along normal geodesics γ. To obtain the desired formulas, we proceed as follows. Let $u\in T_m^{\perp}P$ be a unit vector and denote by $\gamma : r\mapsto\gamma(r)=\exp_\nu(ru)$ the normal geodesic passing through $m=\gamma(0)$. Furthermore, we specialize the frame field $\{E_1,\dots,E_n\}$ such that $E_n(m)=\gamma'(0)=u$. Next, let $\{F_1,\dots,F_n\}$ be the frame field along γ obtained by parallel transport of $\{E_1(m),\dots,E_n(m)\}$. Moreover, let Y_α, $\alpha=1,\dots,n-1$, denote the Jacobi vector fields along γ satisfying the initial conditions

$$\text{(2.2)}\begin{cases} Y_i(0)=E_i(m), & Y_i'(0)=\left(\nabla_{\gamma'}\frac{\partial}{\partial x^i}\right)(m), \quad i=1,\dots,q,\\ Y_a(0)=0, & Y_a'(0)=E_a(m), \qquad\qquad a=q+1,\dots,n-1 \end{cases}$$

where the prime denotes covariant differentiation along γ. Then we have

$$Y_i(r)=\frac{\partial}{\partial x^i}(\gamma(r)),\qquad Y_a(r)=r\frac{\partial}{\partial x^a}(\gamma(r)).$$

Now, put

$$Y_\alpha(r)=D_u(r)F_\alpha(\gamma(r)),\qquad \alpha=1,\dots,n-1.$$

Then the Jacobi equation yields

$$\text{(2.3)}\qquad D_u''+R\circ D_u=0$$

where $R(r)X = R_{\gamma'(r)X}\gamma'(r) \in \{\gamma'(r)\}^{\perp}$. To obtain the initial conditions for the endomorphism field D_u, we use the following well-known Gauss and Weingarten equations for P:

$$\nabla_X Y = \tilde{\nabla}_X Y + T_X Y,$$
$$\nabla_X \xi = T(\xi)X + \nabla^{\perp}_X \xi$$

where X, Y are tangent to P and where ξ is a (local) normal vector field on P. $\tilde{\nabla}$ denotes the Levi Civita connection of the induced metric on P, T is the second fundamental form, $T(\xi)$ the shape operator with respect to ξ and $\nabla^{\perp}$ the normal connection. T and $T(\xi)$ are related by $g(T(\xi)X, Y) = -g(T_X Y, \xi)$ for all X, Y tangent to P. Then, using the initial conditions (2.2), we obtain

$$(2.4) \qquad D_u(0) = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix}, \qquad D'_u(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\perp(u) & I_{n-q-1} \end{pmatrix}$$

where

$$T(u)_{ij} = g(T(u)E_i, E_j)(m),$$
$$\perp(u)_{ia} = g(\perp_{E_i} E_a, E_n)(m)$$

and where $\perp$ is an operator defined in [11] which satisfies $(\perp_X N)(m) = (\nabla^{\perp}_X N)(m)$.

Next, using the generalized Gauss lemma (see [10],[11]), we obtain

$$(2.5) \qquad g_{nn}(p) = 1, \qquad g_{\alpha n}(p) = 0, \qquad \alpha = 1, \ldots, n-1,$$

and furthermore, from the formulas given above, we get for $p = \exp_{\nu}(ru)$, $u \in T^{\perp}_m P$, $\|u\| = 1$:

$$(2.6) \qquad \begin{cases} g_{ij}(p) &= ({}^tD_u D_u)_{ij}(r), \\ g_{ia}(p) &= \dfrac{1}{r}({}^tD_u D_u)_{ia}(r), \\ g_{ab}(p) &= \dfrac{1}{r^2}({}^tD_u D_u)_{ab}(r) \end{cases}$$

for i, $j = 1,\ldots,q$; a, $b = q+1,\ldots,n-1$. Further, we often identify the spaces $\{\gamma'(r)\}^{\perp}$ along γ by means of the parallel basis $\{F_1,\ldots,F_n\}$. Then, using (2.3) , (2.4) and (2.6), we get the following power series expansions for the components g_{ij}, g_{ia} and g_{ab}:

$$\left\{\begin{array}{l} g_{ij}(p) = g(E_i, E_j)(m) + 2rg(T(u)E_i, E_j)(m) + O(r^2), \\ g_{ia}(p) = -rg({}^t \perp (u)E_i, E_a)(m) - \dfrac{2}{3}r^2 g(R(u)E_i, E_a)(m) \\ \qquad +O(r^3), \\ g_{ab}(p) = g(E_a, E_b)(m) - \dfrac{1}{3}r^2 g(R(u)E_a, E_b)(m) + O(r^3) \end{array}\right. \tag{2.7}$$

where $R(u) = R_u.u$ (this will also be denoted by $R(m)$).

Finally, using these Fermi coordinates, we have for a geodesic transformation φ_P:

$$\left\{\begin{array}{l} (\varphi_P^* g)_{ij}(p) = g_{ij}(\varphi_P(p)), \\ (\varphi_P^* g)_{ia}(p) = \rho g_{ia}(\varphi_P(p)), \\ (\varphi_P^* g)_{ab}(p) = \rho^2 g_{ab}(\varphi_P(p)), \\ (\varphi_P^* g)_{nn}(p) = (\rho' r + \rho)^2 g_{nn}(\varphi_P(p)) = (\rho' r + \rho)^2 \end{array}\right. \tag{2.8}$$

for $i, j = 1,\ldots,q$; $a, b = q+1,\ldots,n-1$ and where $s' = \rho' r + \rho$.

3. ISOMETRIC AND CONFORMAL GEODESIC TRANSFORMATIONS

Let P be as in Section 2 and denote by φ_P a geodesic transformation with respect to P. Since φ_P is conformal if and only if $\varphi_P^* g = e^{2\sigma} g$ for some function σ, we get from (2.1) and (2.8):

Proposition 3.1. *φ_P is conformal if and only if*

$$\left\{\begin{array}{l} e^{2\sigma} g_{ij}(p) = g_{ij}(\varphi_P(p)), \\ e^{2\sigma} g_{ia}(p) = \rho g_{ia}(\varphi_P(p)), \\ e^{2\sigma} g_{ab}(p) = \rho^2 g_{ab}(\varphi_P(p)) \end{array}\right. \tag{3.1}$$

with

$$e^{2\sigma} = (s'(r))^2 = (\rho' r + \rho)^2 \tag{3.2}$$

and for $i,\ j = 1,\ldots,q;\ a,\ b = q+1,\ldots,n-1$.

Note that this implies that the function σ only depends on the normal distance r to the submanifold P. Furthermore, by replacing $e^{2\sigma}$ by $(s'(r))^2$ in (3.1), the conditions may be seen as a system of determining differential equations for conformal geodesic transformations.

Next, we first consider the special case of isometric and homothetic geodesic transformations. Here, we have

Proposition 3.2. *A conformal geodesic transformation is homothetic if and only if it is a Euclidean similarity, that is,* $s(r) = Cr$, $C \neq 0$. *Moreover,* φ_P *is a local reflection or the identity map if and only if it is an isometry.*

Proof. Let φ_P be homothetic. Then $e^{2\sigma} = C^2$ for some $C \in \mathbb{R} - \{0\}$. Then (3.2) yields $ds = \pm C dr$ and since $s(0) = 0$, we get $s(r) = \pm Cr$. The isometry occurs if and only if $C^2 = 1$, that is $s(r) = \pm r$.

The converse is immediate. □

Furthermore, we have

Proposition 3.3. *Let* P *be a submanifold with* $dimP \geq 1$. *If* φ_P *is conformal, then* $s'(0) = -1$ *unless it is the identity.*

Proof. Since $\dim P \geq 1$, from (3.1) and (3.2) we get

$$s'(r)^2 g_{ij}\left(\exp_\nu(ru)\right) = g_{ij}\left(\exp_\nu(s(r)u)\right). \tag{3.3}$$

Now, take the limit of both sides in this relation for $r \to 0$. Using (2.7), we get $s'(0)^2 = 1$. Next, we show that $s'(0) = 1$ implies that φ_P is the identity map. (*Note that this last result also holds when* P *is a point.*) To do this, we first treat the case $\operatorname{codim} P > 1$ and consider the condition

$$e^{2\sigma} g_{ab}\left(\exp_\nu(ru)\right) = \rho^2 g_{ab}\left(\exp_\nu(s(r)u)\right) \tag{3.4}$$

which we write as

$$r^2 s'(r)^2 g_{ab}\left(\exp_\nu(ru)\right) = s(r)^2 g_{ab}\left(\exp_\nu(s(r)u)\right). \tag{3.5}$$

Then, it is easily seen by using the expansions (2.7) and putting

$$s(r) = r + \beta_k r^k + O(r^{k+1}), \tag{3.6}$$

that $s''(0) = 0$. So we may suppose $k \geq 3$ for the first non-zero coefficient β_k in (3.6). Putting

$$g_{ab}\left(\exp_\nu(ru)\right) = \sum_{l\geq 0} \alpha_l(m, u, a, b) r^l, \tag{3.7}$$

substitution of (3.6) and (3.7) in (3.5) yields

$$\begin{aligned} \delta_{ab} &+ \sum_{l=1}^{k-2} \alpha_l(m,u,a,b)r^l + \left(2k\beta_k\delta_{ab} + \alpha_{k-1}(m,u,a,b)\right) r^{k-1} + O(r^k) \\ &= \delta_{ab} + \sum_{l=1}^{k-2} \alpha_l(m,u,a,b)r^l + \left(2\beta_k\delta_{ab} + \alpha_{k-1}(m,u,a,b)\right) r^{k-1} + O(r^k). \end{aligned}$$

Comparing the terms of degree $k-1$ yields $\beta_k = 0$ and since s is analytic, we get the required result.

For codim$P = 1$ we may follow a similar procedure by using (3.3) to get the same result. □

It is to expect that the existence of a conformal geodesic transformation φ_P will influence the extrinsic geometry of the submanifold P. The next theorem shows that this indeed true.

Theorem 3.1 *Let P be a submanifold and φ_P a non-trivial conformal geodesic transformation. Then P is totally umbilical and moreover, totally geodesic when codim$P > 1$.*

Proof. Using Proposition 3.3 we put $s(r) = -r + \frac{1}{2}s''(0)r^2 + O(r^3)$ and substitute this in (3.3), making also use of (2.7). This gives

$$\delta_{ij} + 2\left(T(u)_{ij} - s''(0)\delta_{ij}\right) r + O(r^2) = \delta_{ij} - 2T(u)_{ij} r + O(r^2)$$

and hence,

$$T(u) = \frac{1}{2}s''(0)I. \tag{3.8}$$

This means that P is totally umbilical.

Next, suppose codim$P > 1$. Then we proceed in a similar way with (3.5) to get

$$\delta_{ab} - 2s''(0)\delta_{ab}r + O(r^2) = \delta_{ab} - s''(0)\delta_{ab}r + O(r^2).$$

From this we get $s''(0) = 0$ and hence, $T = 0$ which means that P is totally geodesic. □

Corollary 3.1. *Let P be a hypersurface such that φ_P is a conformal geodesic transformation. Then P is a totally umbilical hypersurface with constant mean curvature.*

Proof. This follows at once from (3.8) since the mean curvature h equals $\frac{1}{2}(n-1)s''(0)$ at each point $m \in P$. □

Now, we prove the main result of this section.

Theorem 3.2. *Let P be a q-dimensional submanifold of (M, g) of dimension n. If $0 < q < n-1$, then each conformal geodesic transformation is an isometry.*

Proof. Again, Proposition 3.3 implies that it suffices to put

$$s(r) = -r + \sum_{k\geq 1} \beta_k r^k.$$

Here we may suppose $k > 2$ since the result in Theorem 3.1 gives $\beta_2 = 0$. So, we write

$$s(r) = -r + \beta_k r^k + O(r^{k+1})$$

where β_k is the first non-zero coefficient for $k \geq 1$. Thus $k \geq 3$. Next, put

$$g_{ij}\left(\exp_\nu(ru)\right) = \delta_{ij} + \sum_{k\geq 1} \alpha_k(m, u, i, j) r^k. \tag{3.9}$$

Now, we substitute again in (3.3) to get

$$\delta_{ij} + \sum_{l=1}^{k-2} \alpha_l(m,u,i,j) r^l + \left(\alpha_{k-1}(m,u,i,j) - 2k\beta_k\delta_{ij}\right) r^{k-1} + O(r^k)$$
$$= \delta_{ij} + \sum_{l=1}^{k-2} (-1)^l \alpha_l(m,u,i,j) r^l + (-1)^{k-1}\alpha_{k-1}(m,u,i,j) r^{k-1}$$
$$+ O(r^k).$$

So, we have

$$2k\beta_k\delta_{ij} = \left(1 - (-1)^{k-1}\right) \alpha_{k-1}(m,u,i,j) \tag{3.10}$$

from which it follows that $\beta_k = 0$ if k is odd. Next, let k be even, say $k = 2t$. Then we get from (3.10):

$$\beta_{2t} = \frac{1}{2t}\alpha_{2t-1}(m,u,i,i) \tag{3.11}$$

for some $i \in \{1, \dots, q\}$. Moreover, since $\operatorname{codim} M > 1$, there exits a unit vector $v \in T_m^\perp P$ which is orthogonal to u. Then, from (3.11) we obtain

$$2t\beta_{2t} = \alpha_{2t-1}(m,u,i,i) = \alpha_{2t-1}(m, u\cos\theta + v\sin\theta, i, i)$$

and by taking the limit for $\theta \to \pi$, we get

$$\alpha_{2t-1}(m,u,i,i) = \alpha_{2t-1}(m,-u,i,i).$$

Since $\alpha_{2t-1}(m,u,i,i) = -\alpha_{2t-1}(m,u,i,i)$, we finally get $\alpha_{2t-1}(m,u,i,i) = 0$ and so $\beta_{2t} = 0$.

Proceeding by induction, we find $s(r) = -r$ and then the result follows from Proposition 3.2. □

It follows from this theorem that the only interesting cases for non-trivial conformal φ_P are the cases when P is a point or a hypersurface. These φ_P will be considered in Section 4 and Section 5, respectively. The isometric case is treated in [5].

From now on we shall only consider *non-trivial* conformal geodesic transformations without mentioning it explicitly.

4. CONFORMAL GEODESIC TRANSFORMATIONS WITH RESPECT TO POINTS

We begin by writing down the necessary and sufficient conditions for φ_P to be conformal when P is a point m. We use (3.5) with $a, b = 1, \ldots, n-1$, (3.7) with $\alpha_0(m,u,a,b) = \delta_{ab}$ and put

$$s(r) = \sum_{k\geq 1} \beta_k r^k.$$

Then, substitution yields at once

Proposition 4.1. *A geodesic transformation φ_m with respect to a point $m \in (M,g)$ is conformal if and only if*

$$\begin{aligned}
\beta_1^2\left(1-\beta_1^k\right)\alpha_k(m,u,a,b) &= \delta_{ab}\left(\sum_{p+q=k+2}(1-pq)\beta_p\beta_q\right)\\
&\quad +\beta_1^2\sum_{l=1}^{k-1}\alpha_l(m,u,a,b)\left(\sum_{p_1+\cdots+p_l=k}\beta_{p_1}\ldots\beta_{p_l}\right)\\
&\quad -\sum_{l=1}^{k-1}\alpha_{k-l}(m,u,a,b)\left(\sum_{p+q=l+2}pq\beta_p\beta_q\right)\\
&\quad +\sum_{l=1}^{k-1}\left(\sum_{p+q=l+2}\beta_p\beta_q\right)\\
&\qquad \left(\sum_{\nu\geq 1}\alpha_\nu(m,u,a,b)\left(\sum_{p_1+\cdots+p_\nu=k-l}\beta_{p_1}\ldots\beta_{p_\nu}\right)\right)
\end{aligned} \tag{4.1}$$

for all $a, b = 1, \ldots, n-1$ and $k \geq 1$.

Here, we note that, in contrast to the case of $\dim P \geq 1$, the conditions (3.5) do not yield initial conditions for the solutions but we have

Proposition 4.2. *Let φ_m be a geodesic transformation with respect to a point m. Then the function s satisfies $s'(0)^2 = 1$ if and only if φ_m is the identity map or a local reflection.*

Proof. Clearly $s'(0)^2 = 1$ for the identity map or a local reflection. Conversely, suppose $s'(0) = 1$. Then φ_m is the identity map (see the proof of Proposition 3.3. So, it remains to consider the case $s'(0) = -1$. Put

$$s(r) = -r + \beta_k r^k + O(r^{k+1})$$

where β_k denotes the second non-zero term in the expansion. Substitution in (3.5) then yields $\beta_2 = 0$ and for $k \geq 3$ we have

$$\begin{aligned}\delta_{ab} &+ \sum_{l=1}^{k-2} \alpha_l(m,u,a,b)r^l + \{\alpha_{k-1}(m,u,a,b) - 2k\beta_k\delta_{ab}\} + O(r^k) \\ &= \delta_{ab} + \sum_{l=1}^{k-2}(-1)^l\alpha_l(m,u,a,b)r^l \\ &\qquad + \left\{-2\beta_k\delta_{ab} + (-1)^{k-1}\alpha_{k-1}(m,u,a,b)\right\} r^{k-1} + O(r^k).\end{aligned}$$

Comparing the coefficients of r^{k-1} yields $\beta_k = 0$ for an odd k. Furthermore, when k is even, say $k = 2t$, we get

$$\beta_{2t} = \frac{1}{2t-1}\delta_{ab}\alpha_{2t-1}(m,u,a,b).$$

Proceeding as in the proof of Theorem 3.2 yields $\beta_{2t} = 0$. An induction procedure then implies $\beta_k = 0$ for all $k \in \mathbb{N}_0$ and hence $s(r) = -r$, which gives the required result. □

Now, we shall write down a form for the necessary and sufficient existence conditions by using the Jacobi operator $R(m) = R_u.u$ and its covariant derivatives $R^{(k)}(m) = (\nabla^k_{u\ldots u}R)_u.u$. We have

Theorem 4.1. *Let $m \in (M,g)$. Then there exists a non-isometric, conformal geodesic transformation φ_m with respect to m if and only if*

$$\text{(4.2)} \qquad \begin{cases} R^{(2k)}(m) = F_k(m)I, \\ R^{(2k+1)}(m) = 0 \end{cases}$$

for all $k \in \mathbb{N}$ and where $F_k(m)$ is a constant (independent of u).

Proof. First, suppose (4.2) is satisfied. We shall prove that in this case we have

$$\text{(4.3)} \qquad g_{ab}(\exp_m(ru)) = f(r)\delta_{ab}$$

and then, by using (3.5), it follows that there exists a conformal φ_P for each value $s'(0)$ where $s'(0)^2 \neq 1$. To prove (4.3), we use (2.6) for g_{ab}. From the Jacobi equation (2.3), we obtain the recursion formula

$$D_u^{(l+2)}(0) = -\sum_{k=0}^{l} C_l^k R^{(l-k)}(m) D_u^{(k)}(0) \tag{4.4}$$

with the initial conditions $D_u(0) = 0$, $D'_u(0) = I$. Then, it follows from (4.2) that $D_u^{(l)}(0) = \lambda_l I$ for $l \geq 0$ and this implies, with (2.6), the required formula (4.3).

Conversely, let φ_m be a non-isometric, conformal geodesic transformation. Then, by an induction procedure we obtain from (4.1)

$$\begin{cases} \alpha_k(m,u,a,b) = 0, & a \neq b \\ \alpha_k(m,u,a,a) = \alpha_k(m,u,b,b) = \alpha_k(m) \end{cases} \tag{4.5}$$

for a, $b = 1, \ldots, n-1$ and $k \in \mathbb{N}$ and so, we have

$$g_{ab}\left(\exp_m(ru)\right) = \delta_{ab}\left(1 + \sum_{k \geq 1} \alpha_k(m) r^k\right).$$

Next, using again (2.6) for g_{ab}, we get

$$\begin{aligned} (k+2)!\alpha_{k-2}(m,u,a,b) &= ({}^tD_u D_u)_{ab}^{(k)}(0) \\ &= k\left({}^tD_u^{(k-1)}(0) + D_u^{(k-1)}(0)\right) + \sum_{l=2}^{k-2} C_{k-l}^k D_u^l(0) D_u^{(k-l)}(0). \end{aligned} \tag{4.6}$$

Now, by an induction procedure, we obtain that $D_u^{(t)}(0) = \lambda_t I$. Indeed, let $D_u^{(k)}(0) = \lambda_k I$ for $k < l$. Then, it follows from (4.4) that $D_u^{(l)}(0)$ is symmetric and this, together with (4.5) and (4.6) implies the result. From this and (4.4) it then follows that $R^{(k)}(m) = F_k(m)I$ where $F_k(m)$ does not depend on the vector u. Finally, this yields $R^{(2k+1)}(m) = 0$. □

Remark 4.1. Note that, when (4.2) holds, then the geodesic symmetry centered at m is automatically isometric since $R^{(2k+1)}(m) = 0$ for all $k \geq 0$ [5]. However, the converse does not hold. Indeed, for any locally symmetric space of non-constant curvature, the geodesic symmetries are isometric but (4.2) does not hold for $k = 0$. This shows that the existence for a non-isometric conformal geodesic transformation is more restrictive than the existence of an isometric geodesic symmetry.

As a consequence of Theorem 4.1 we obtain the following extension of a result proved in [6],[15].

Theorem 4.2. *Let (M, g) be a Riemannian manifold which admits a conformal geodesic transformation with respect to each point. Then (M, g) is locally symmetric and moreover, (M, g) is a space of constant curvature if and only if there also exists a non-isometric conformal geodesic transformation with respect to some point.*

Proof. If φ_m is an isometry, then we clearly have $R^{(2k+1)}(m) = 0$ for $k \geq 0$. The same result follows from (4.2) when φ_m is non-isometric. This proves the first part of the theorem. Moreover, since in that case (M, g) is locally homogeneous, the rest follows from (4.2) for $k = 0$. □

Finally, we have

Theorem 4.3. *Let (M, g) be a locally symmetric space which admits a conformal geodesic transformation φ_m with respect to a point $m \in M$. Then*

(i) *φ_m is a local reflection, or*

(ii.a) *φ_m is a Euclidean similarity, that is, $s(r) = Cr$, $C^2 \neq 0, 1$, if and only if (M, g) is locally flat;*

(ii.b) *φ_m is a non-Euclidean similarity defined by*

$$\tan s\frac{\sqrt{c}}{2} = C\tan r\frac{\sqrt{c}}{2}, \qquad C^2 \neq 0, 1,$$

if and only if (M, g) is a space of constant curvature $c > 0$;

(ii.c) *φ_m is a non-Euclidean similarity defined by*

$$\tanh s\frac{\sqrt{c}}{2} = C\tanh r\frac{\sqrt{c}}{2}, \qquad C^2 \neq 0, 1,$$

if and only if (M, g) *is a space of constant sectional curvature* $-c < 0$.

Proof. If φ_m is non-isometric, then (M, g) is a space of constant curvature. The rest follows from the explicit solution of (3.5). (See also [7].)

5. CONFORMAL GEODESIC TRANSFORMATIONS WITH RESPECT TO HYPERSURFACES

Now, we consider the case of a hypersurface P. Then the defining equations are (3.3). Using now the power series expansions

$$s(r) = \sum_{k\geq 1} \beta_k r^k , \qquad \beta_1 = -1,$$

and (3.9), we get by substitution in (3.3):

Proposition 5.1. *A geodesic transformation* φ_P *with respect to a hypersurface is conformal if and only if*

$$\begin{aligned} 2(k+1)\beta_{k+1}\delta_{ij} &= \left(1-(-1)^k\right)\alpha_k(m,u,i,j) + \delta_{ij} \sum_{\substack{p+q=k+2 \\ p,q>1}} pq\beta_p\beta_q \\ &+ \sum_{l=1}^{k-1} \alpha_l(m,u,i,j)\left(\sum_{p+q=k-l+2} pq\beta_p\beta_q - \sum_{p_1+\cdots+p_t=k} \beta_{p_1}\ldots\beta_{p_t}\right) \end{aligned} \tag{5.1}$$

for $i,\ j = 1, \ldots, n-1$ *and* $k \in \mathbb{N}$.

Note that this proposition implies that there exists at most one nontrivial conformal geodesic transformation with respect to P.

Furthermore, it follows from Corollary (3.1) that P is a totally umbilical hypersurface. Now, we prove

Proposition 5.2. *Let* P *be a totally umbilical hypersurface which is not totally geodesic. If there exists a conformal geodesic transformation with respect to* P*, then the components* $g_{ij}\left(exp_\nu(ru)\right)$ *are independent of* $m \in P$.

Proof. From (3.8) we get $s''(0) = 2(n-1)^{-1}\mathrm{tr}T(u)$ and hence $s''(0) \neq 0$. This and (2.7) shows that $\alpha_1(m,u,i,j)$ does not depend on m. Now, we proceed by induction. First, let $\alpha_1(m,u,i,j)$, ..., $\alpha_{2k}(m,u,i,j)$ be independent of m. Then (5.1) shows that $\alpha_{2k+1}(m,u,i,j)$ is also independent of m. Next, let $\alpha_1(m,u,i,j)$, ..., $\alpha_{2k-1}(m,u,i,j)$ be constant on P. Considering (5.1) for β_{2k+2} and β_{2k+3} and using the condition $\beta_3 = -\beta_2^2 \neq 0$ obtained from (5.1), one finds an expression for $\alpha_{2k}(m,u,i,j)$ in terms of $\alpha_l(m,u,i,j)$, $l < 2k$, and some β_l. So, the result follows. □

Note that the converse of this result does not necessarily hold since (3.3) gives a system of differential equations which may have no solution.

Now, we shall use the Jacobi operator $R(m)$ and its covariant derivatives to express the necessary and sufficient existence conditions.

Theorem 5.1. *Let P be a totally umbilical hypersurface which is not totally geodesic. Then there exists a conformal geodesic transformation with respect to P if and only if*

$$R^{(k)}(m) = F_k(u)I \tag{5.2}$$

for all $k \geq 0$, where F is independent of m.

Proof. Assume (5.2) holds. We shall show that $g_{ij}(\exp_\nu(ru)) = f(r)^2\delta_{ij}$. The existence then follows from (3.3). As in Section 4 we consider the expressions (2.6) and the solutions of the Jacobi equation (2.3) with initial values $D_u(0) = I$, $D'_u(0) = T(u)$. First, from (2.3) we again get

$$D_u^{(k+2)}(0) = -\sum_{l=0}^{k} C_k^l R^{(k-l)}(m) D_u^l(0). \tag{5.3}$$

From this, (5.2) and the umbilicity it follows that $D_u^{(l)}(0) = \lambda_l I$ and hence $D_u = f(r)I$. This yields the required result.

Conversely, let φ_P be a conformal geodesic transformation. Then it follows from Proposition (5.2) that the coefficients in the power series expansion of g_{ij} are independent of m and proceeding as in the proof of

Proposition (5.2), one obtains by means of an induction procedure:

$$\text{(5.4)} \qquad \begin{cases} \alpha_k(u,i,j) = 0, & i \neq j, \\ \alpha_k(u,i,i) = \alpha_k(u,j,j) \end{cases}$$

for all $k \geq 0$. Furthermore, proceeding as in the proof of Theorem (4.1), we obtain $D_u(r) = f(r)I$ and then (5.2) follows by using (5.3). □

To obtain the above result we needed $s''(0) \neq 0$, that is, P is not totally geodesic. Now, we shall show that (5.2) still gives necessary and sufficient conditions for $s''(0) = 0$ under an additional condition. More precisely, we have

Theorem 5.2. *Let P be a totally geodesic hypersurface such that*

$$\text{(5.5)} \qquad R^{(k)}(m) = F_k(u)I, \qquad k \geq 0.$$

Then there exists a conformal geodesic transformation φ_P. Moreover, if R' does not vanish identically on P, then the conditions (5.5) are also necessary conditions for the existence of a conformal φ_P.

Proof. The sufficiency of the conditions (5.5) follows as in the proof of Theorem 5.1.

To prove that they are necessary, it is enough to show that the coefficients $\alpha_k(m,u,i,j)$ are independent of m and satisfy (5.4). To do this, we proceed by induction. First, suppose (5.4) holds for all $l \leq 2k$. Then it also holds for $\alpha_{2k+1}(m,u,i,j)$ as follows from (5.1). So, it remains to prove the assertion that (5.4) holds for $\alpha_{2k}(m,u,i,j)$ when it holds for $\alpha_l(m,u,i,j)$ with $l \leq 2k-1$. Note that in this case $\beta_3 = -\beta_2^2 = 0$. Next, it follows from (5.1) that $4\beta_4\delta_{ij} = \alpha_3(m,u,i,j)$ and since $3\alpha_3(m,u,i,j) = -g(R'(m)E_i, E_j)$, our hypothesis implies $\beta_4 \neq 0$. Further, (5.1) yields $\beta_7 = -2\beta_4^2$. Now, considering the expressions (5.1) for β_{2k+4}, β_{2k+5} and β_{2k+7}, it follows that $\alpha_{2k+1}(m,u,i,j)$ and furthermore $\alpha_{2k}(m,u,i,j)$, satisfy (5.4). So, this completes the proof. □

Remark 5.1. It is proved in [5] that a local reflection φ_P with respect to P is isometric if and only if

(i) P is totally geodesic;

(ii) $R^{(2k+1)}(m) = 0$ for all $k \geq 0$ and all $m \in P$.

Hence, it follows that the geodesic transformations in Theorem 5.1 and Theorem 5.2 are different from a local reflection.

We do not know if the additional condition for R' in Theorem 5.2 can be deleted.

For a locally symmetric space (M, g) we have

Theorem 5.3. *Let P be a hypersurface in a locally symmetric space of dimension ≥ 3. If there exists a conformal geodesic transformation φ_P with respect to P, then P is totally umbilical and moreover, totally geodesic if and only if φ_P is a reflection.*

Proof. For locally symmetric spaces, (ii) in Remark 5.1 is satisfied and hence a local reflection is an isometry if and only if P is totally geodesic. The result follows from the fact that there can only exist at most one conformal φ_P. □

The existence of non-isometric conformal geodesic transformations with respect to a hypersurface in a locally symmetric space influences highly the geometry of the ambient space. Indeed, we have

Theorem 5.4. *Let P be a hypersurface in a locally symmetric space (M, g) of dimension ≥ 4 such that there exists a non-isometric, conformal geodesic transformation φ_P. Then (M, g) has constant curvature.*

Proof. Since φ_P is non-isometric, Theorem 5.3 implies that P is totally umbilical but not totally geodesic. Now, in [4] it is shown that a locally symmetric space which admits such a hypersurface is locally conformally flat for $\dim M \geq 4$ and hence, one of the following spaces: a real space form, a local product of two real space forms $M(c) \times \overline{M}(-c)$ or a local product $\mathbb{R} \times M(c)$. But it follows from (5.2) that $R(m)$ has only one eigenvalue and then it is easy to check that we can only have the first case. □

Finally, using the classification of totally umbilical hypersurfaces in real space forms (see, for example, [3]), we have

Theorem 5.5. *Let (M, g) be a locally symmetric space and P a hypersurface such that φ_P is a non-isometric conformal transformation. Then we have*

(i) (M, g) is locally flat, P is an open part of a geodesic sphere of radius α and φ_P is the Euclidean inversion determined by

$$(s+\alpha)(r+\alpha) = \alpha^2, \quad \textit{or}$$

(ii) (M, g) is a space of constant curvature $c > 0$, P is an open part of a geodesic sphere of radius α and φ_P is the non-Euclidean inversion determined by

$$\tan(s+\alpha)\frac{\sqrt{c}}{2}\tan(r+\alpha)\frac{\sqrt{c}}{2} = \left(\tan\alpha\frac{\sqrt{c}}{2}\right)^2, \quad \textit{or}$$

(iii) (M, g) is a space of constant curvature $-c < 0$ and

(iii.a) P is an open part of a geodesic sphere of radius α and φ_P is the non-Euclidean inversion determined by

$$\tanh(s+\alpha)\frac{\sqrt{c}}{2}\tanh(r+\alpha)\frac{\sqrt{c}}{2} = \left(\tanh\alpha\frac{\sqrt{c}}{2}\right)^2, \quad \textit{or}$$

(iii.b) P is an open part of a tube of radius α about a totally geodesic hypersurface and φ_P is defined by

$$\arctan\sinh(s+\alpha)\sqrt{c} + \arctan\sinh(r+\alpha)\sqrt{c} = 2\arctan\sinh\alpha\sqrt{c},$$

or

(iii.c) P is an open part of a horosphere and φ_P is determined by

$$e^{s\sqrt{c}} + e^{r\sqrt{c}} = 2.$$

Proof. (*i*), (*ii*) and (*iii.a*) follow at once by solving the differential equation (3.3) as has been done in [7]. For (*iii.b*) and (*iii.c*) one proceeds in a similar way by taking into account that the shape operator $T(u)$ is given by

$$T(u) = \left(\sqrt{c}\tanh \alpha\sqrt{c}\right) I$$

for $(iii.b)$ and by

$$T(u) = \sqrt{c}I$$

for $(iii.c)$. In each of these cases one starts by computing explicitly D_u, then determines g_{ij} and finally, solves (3.3). □

6. GEODESIC TRANSFORMATIONS ON WARPED PRODUCTS

If (M, g) is a Riemannian manifold equipped with a family of submanifolds, one may expect that the existence of special geodesic transformations with respect to all elements of this family will influence not only the nature of these submanifolds but also put restrictions on the geometry of (M, g). For geodesic reflections this has been investigated in the framework of contact and flow geometry and also for Riemannian foliations. We refer to [2], [10], [16] for some results and for further references.

In this section we shall consider a similar situation for *warped products*. This is quite natural since the fibers of such a product are always totally umbilical submanifolds. First, we recall some basic facts and refer to [1], [13] for more details. Let $M = N \times_f F$ be a warped product of two Riemannian manifolds (N, g_N) and (F, g_F) with warping function $f : N \to \mathbb{R}^+$ and metric

$$g = \pi_1^* g_N + (f \circ \pi_1)^2 \pi_2^* g_F$$

where $\pi_1 : N \times F \to N$ and $\pi_2 : N \times F \to F$ are the projections. The leaves $\mathcal{N} = N \times \{\pi_2(m)\} = \pi_1^{-1}\pi_2(m)$ and the fibers $\mathcal{F} = \{\pi_1(m)\} \times F = \pi_2^{-1}\pi_1(m)$ for $m \in M$ form two classes of submanifolds. The leaves are totally geodesic and the fibers are totally umbilical submanifolds with shape operator defined by

$$T(X) = \frac{1}{f} g(\operatorname{grad} f, X) I. \tag{6.1}$$

First, we prove

Theorem 6.1. *Let $M = N \times_f F$ be a warped product and let $\varphi_{\mathcal{F}}$ be a geodesic transformation with respect to the fiber $\mathcal{F}$ passing through $m \in M$. For* $\dim M > 1$, *$\varphi_{\mathcal{F}}$ is a conformal geodesic transformation if and only if it is an isometry. This holds for any $m \in M$ if and only if M is a direct product $N \times_f F$ ($f = const.$) where N is locally symmetric.*

Proof. Let $\varphi_{\mathcal{F}}$ be a conformal geodesic transformation with respect to $\mathcal{F}$. Since $\operatorname{codim}\mathcal{F} > 1$ it follows from Theorem 3.2 that $\varphi_{\mathcal{F}}$ is an isometry and since we only consider non-trivial $\varphi_{\mathcal{F}}$, it must be a local reflection. In [5], it is shown that a reflection $\varphi_{\mathcal{F}}$ is an isometry if and only if

(i) $\mathcal{F}$ is totally geodesic;

(ii) $\left(\nabla^{(2k)}_{u\ldots u}\right)_{uv} u$ is normal to $\mathcal{F}$;

(iii) $\left(\nabla^{(2k+1)}_{u\ldots u}\right)_{uv} u$ is tangent to $\mathcal{F}$;

(iv) $\left(\nabla^{(2k+1)}_{u\ldots u}\right)_{ux} u$ is normal to $\mathcal{F}$

where u, v are normal and x tangent to $\mathcal{F}$. So, $\mathcal{F}$ has to be totally geodesic. If this holds for all $m \in M$, it follows from (6.1) that $\mathcal{F}$ has to be a constant function and hence, M is a direct product. Furthermore, (iii) implies $\nabla_u R_{uvuv} = 0$ and this yields at once that N is locally symmetric.

Conversely, if M is a direct product $N \times_f F$, $f =$ const. and N symmetric, (i) – (iv) are satisfied and so each reflection with respect to any fiber $\mathcal{F}$ is isometric. □

It follows that the interesting case occurs when N is one-dimensional. Then we have

Theorem 6.2. *Let $M = (-\varepsilon, \varepsilon) \times_f F$ be a warped product with warping function $f : r \mapsto f(r)$. Then the conformal geodesic transformations with respect to the fiber $\mathcal{F}$ through m corresponding to $r = 0$ are determined by*

$$f(s)^{-1} ds = f(r)^{-1} dr$$

Proof. Since the fibers are hypersurfaces, it follows from Proposition 3.1 that we have to consider (3.3) and hence to determine $g_{ij}(\exp_\nu(ru))$. We do this by using the method described in Section 2.

First, the shape operator T of $\mathcal{F}$ is given by $T = (\ln f)'(0)I$ and furthermore, the Jacobi operator at $p = \exp_\nu(ru)$ is given by [13]

$$R(u) = -\frac{f''(r)}{f(r)}I.$$

It follows that the endomorphism field D is given by

$$D = \{z(r) + (\ln f)'(0)y(r)\}\, I$$

where z and y are solutions of

$$x'' - \frac{f''}{f}x = 0$$

with initial values $z(0) = 1$, $z'(0) = 0$, $y(0) = 0$, $y'(0) = 1$. Hence, we have

$$\begin{aligned} z(r) &= f(0)^{-1}f(r) - f(r)f'(0)\int_0^r f(t)^{-2}dt, \\ y(r) &= f(0)f(r)\int_0^r f(t)^{-2}dt \end{aligned}$$

and so

$$D(r) = f(0)^{-1}f(r)I.$$

This yields

$$g_{ij}(\exp_\nu(ru)) = f(0)^{-2}f(r)^2\delta_{ij}$$

and now the result follows at once from (3.3). □

We finish this section with

Theorem 6.3. *Let (M, g) be a Riemannian manifold equipped with two complementary and orthogonal foliations $\mathcal{N}$ and $\mathcal{F}$ such that the leaves of $\mathcal{N}$ are one-dimensional. Then M is locally isometric to a warped* *product if and only if there exists a geodesic conformal transformation with respect to the leaves of $\mathcal{F}$.*

Proof. If M is locally a warped product, then the result follows from Theorem 6.2.

Conversely, let φ be a conformal geodesic transformation with respect to the leaves of $\mathcal{F}$. If φ is a local reflection, it is an isometry and M is locally a direct product. If φ is non-isometric, then the leaves of $\mathcal{F}$ are totally umbilical hypersurfaces with constant mean curvature (Corollary 3.1), that is, $\mathcal{F}$ is a spherical foliation. Then the result follows from [14]. □

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