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ON THE EXISTENCE OF DEGENERATE HYPERSURFACES IN SASAKIAN MANIFOLDS

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ABSTRACT. We determine necessary and sufficient conditions for the existence of degenerate hypersurfaces, tangent to the structure vector field ξ, in R_s^{2n+1}. The main result is that in R_s^{2n+1} all hypersurfaces tangent to ξ are not degenerate. Finally we construct an example of a degenerate hypersurface, tangent to ξ in R_1^5.

1. INTRODUCTION

Indefinite Kahler manifolds have been introduced by Barros-Romero [1]. Because of the signature of the metric we expect some essential changes in the study of submanifolds in such spaces. It is known that the null hypersurfaces have an important role in the study of various problems from electromagnetism, relativity, and different branches of mathematics (see for example the system determinated by Maxwell's and Einstein's equations) and consequently it is necesary to study these hypersurfaces. The study of indefinite Sasakian manifolds is closely related to the study of indefinite Kähler manifolds. However, the existence of the structure vector field implies some important changes.

The main purpose of the paper is to prove that there do not exist degenerate hypersurfaces in R_n^{2n+1} which are tangent to the structure vector field ξ. In the first section we recall the ϵ-Sasakian structure on R_s^{2n+1} given by Bejancu-Duggal [2]. In order to get our main result

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we obtain in section 2 the necessary and sufficient conditions for a hypersurface of R_s^{2n+1} be degenerate and tangent to ξ (Theorems 2.1 and 2.2). We close the paper with an example of a degenerate hypersurface tangent to ξ in R_1^5.

2. PRELIMINARIES

Let $\tilde{M}$ be a paracompact real $(2n+1)$-dimensional smooth manifold. Denote by $F(\tilde{M})$-the algebra of all real smoth functions on $\tilde{M}$ and by $\Gamma(\tilde{M})$ the $F(\tilde{M})$-module of smooth vector fields on $\tilde{M}$. We use the same notation for the module of sections of an arbitrary vector bundle. Suppose there exists on $\tilde{M}$ a semi-Riemannian metric $\tilde{g}$ of constant index s (cf. O'Neil [11], p. 54). Then, according to Bejancu-Duggal [2], $\tilde{M}$ is called an *(ϵ)-almost contact metric manifold* if there exists a tensor field f of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\begin{aligned} f^2 &= -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \eta\circ f = 0,\\ f(\xi) &= 0, \quad rank\ f = 2n, \end{aligned}$$

and

$$\begin{aligned} \tilde{g}(fX,fY) &= \tilde{g}(X,Y) - \epsilon\eta(X)\eta(Y);\\ \eta(X) &= \epsilon g(X,\xi), \quad \forall\, X,Y\in\Gamma(TM), \end{aligned}$$

where $\epsilon = 1\ or -1$, depending on whether ξ is spacelike or time like, respectively, and I is the identity on the tangent bundle $T\tilde{M}$. In this case (f,ξ,η,g,ϵ) is called an *ϵ-almost contact metric structure.* An (ϵ)-almost contact metric manifold $\tilde{M}$ is said to be an *(ϵ)-Sasakian manifold* if the tensor fields involved in the definition satisfy

$$(\tilde{\nabla}_X f)Y = \tilde{g}(X,Y)\xi - \epsilon\eta(Y)X, \quad \forall\ X,Y\ \in\ \Gamma(T\tilde{M}).$$

As an example, following Bejancu-Duggal [2], we give here the (ϵ)- Sasakian structure of R_s^{2n+1}, $0 < s \leq n$. First we make the following notations:

$0_{k,h}$ is the $k \times h$ null matrix, I_k is the $k \times k$ unit matrix

$$\epsilon^{n+a} = \epsilon^a = \begin{cases} -1 & \text{for } a \in \{1,2,\dots,s\} \\ 1 & \text{for } a \in \{s+1,\dots,n\} \end{cases} \tag{1}$$

We take (x^i, y^i, z), $i = \{1,\dots,n\}$, as cartesian coordinates on R_s^{2n+1}. Then with respect to the natural field of frames $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z}\}$ define the tensor field f of type $(1,1)$ by its matrix

$$[f] = \begin{bmatrix} O_{n,n} & I_n & O_{n,1} \\ -I_n & O_{n,n} & O_{n,1} \\ O_{1,n} & \epsilon^a y^a & 0 \end{bmatrix}.$$

The structure vector field ξ and the 1-form η are defined by

$$\xi = 2\epsilon \frac{\partial}{\partial z}, \tag{2}$$

and

$$\eta = \frac{\epsilon}{2}\{dz - \sum_{i=1}^{n} \epsilon^i y^i dx^i\},$$

respectively. Finally the semi-Riemannian metric $\tilde{g}$ is defined by the matrix

(3)

$$[\tilde{g}] = \frac{1}{4}\begin{bmatrix} -\delta_{ab} + y^a y^b & -y^a y^{b^*} & O_{s,s} & O_{s,n-s} & y^a \\ -y^a y^{b^*} & \delta_{a^*b^*} + y^{a^*} y^{b^*} & O_{n-s,s} & O_{n-s,n-s} & -y^{a^*} \\ O_{s,s} & O_{s,n-s} & -I_s & O_{s,n-s} & O_{s,1} \\ O_{n-s,s} & O_{n-s,n-s} & O_{n-s,s} & I_{n-s} & O_{n-s,1} \\ y^a & -y^{a^*} & O_{1,s} & O_{1,n-s} & 1 \end{bmatrix}$$

where $a, b \in \{1,\dots s\}$, and $a^*, b^* \in \{s+1, \dots, n\}$. Then $(f, \xi, \eta, \tilde{g})$ defines an (ϵ)-Sasakian structure on R_s^{2n+1}.

3. DEGENERATE HYPERSURFACES IN THE SASAKIAN MANIFOLD R_s^{2n+1}

Suppose M is locally given by the equations

$$(4)\qquad \begin{array}{l} x^A = h^A(u^1, u^2, \dots u^{2n}), \ \ \mathrm{rank}[\frac{\partial h^A}{\partial u^\alpha}] = 2n; \\ A \in \{1, \dots, 2n+1\}, \ \ \alpha \in \{1, \dots, 2n\} \end{array}$$

where h^A are smooth functions on a domain $D \subset R^{2n}$. Consider R_s^{2n+1} endowed with the semi-euclidian metric $\tilde{g}$ given by (3). Then the metric tensor field induced by $\tilde{g}$ on M is denoted by g and has the local components

$$g_{\alpha\beta} = \tilde{g}_{AB} \frac{\partial h^A}{\partial u^\alpha} \frac{\partial h^B}{\partial u^\beta}.$$

According to Bejancu-Duggal [3], M is degenerate if and only if $rank[g_{\alpha\beta}] = 2n-1$ on M, i.e.,

$$(5)\qquad \Delta = \det \left| \frac{\partial h^A}{\partial u^\alpha} g_{AB} \frac{\partial h^B}{\partial u^\beta} \right| = 0.$$

Now, from the general theory of determinants, we recall (see [4] pp. 123).

Proposition 2.1. *Let $A = [a_{ij}]$ and $B = [b_{jk}]$ two matrices of type $n \times m$ and $m \times n$, $(n \leq m)$, respectively. Then the determinant of the matrix $C = A \times B$ is given by*

$$\det\ C = \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq m} \begin{vmatrix} a_{1j_1} & a_{2j_1} & \cdots & a_{nj_1} \\ a_{1j_2} & a_{2j_2} & \cdots & a_{nj_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1j_n} & a_{2j_n} & \cdots & a_{nj_n} \end{vmatrix} \times \begin{vmatrix} b_{j_1 1} & b_{j_1 2} & \cdots & b_{j_1 n} \\ b_{j_2 1} & b_{j_2 2} & \cdots & b_{j_2 n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{j_n 1} & b_{j_n 2} & \cdots & b_{j_n n} \end{vmatrix}.$$

In order to calculate Δ from (5) we use the following notation

$$D^A = \begin{vmatrix} \frac{\partial h^1}{\partial u^1} & \cdots \frac{\partial h^{A-1}}{\partial u^1}, & \frac{\partial h^{A+1}}{\partial u^1} \cdots & \frac{\partial h^{2n+1}}{\partial u^1} \\ \frac{\partial h^1}{\partial u^2} & \cdots \frac{\partial h^{A-1}}{\partial u^2}, & \frac{\partial h^{A+1}}{\partial u^2} \cdots & \frac{\partial h^{2n+1}}{\partial u^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h^1}{\partial u^{2n}} & \cdots \frac{\partial h^{A-1}}{\partial u^{2n}}, & \frac{\partial h^{A+1}}{\partial u^{2n}} \cdots & \frac{\partial h^{2n+1}}{\partial u^{2n}} \end{vmatrix}, \quad A \in \{1, \cdots, 2n+1\}.$$

Also, we denote by M_{AB} the determinant of the $2n$ x $2n$ matrix obtained by deleting the Ath row and Bth column from $[\tilde{g}]$ given by (3). As $[\tilde{g}]$ is a symmetric matrix we have $M_{AB} = M_{BA}$. By direct calculation we obtain

$$(6)\quad M_{AB} = \begin{cases} 0, \text{ for } 1 \le A \ne B \le 2n, \text{ or } n+1 \le A \le 2n \\ \qquad \text{and } B = 2n+1 \\ \epsilon^A, \text{ for } A = B \in \{1, \dots, 2n\} \\ \epsilon^A(-1)^A y^A, \ \ for \ \ A \in \{1, \dots, n\} \text{ and } B = 2n+1 \\ 1 + \sum_{a=1}^{n} \epsilon^a (y^a)^2, \text{ for } A = B = 2n+1. \end{cases}$$

Theorem 2.1. *Let M be a hypersurface of the (ϵ)- Sasakian manifold R_s^{2n+1}. Then M is degenerate if and only if*

$$\sum_{a=1}^{n} \{\epsilon^a (D^a + (-1)^a y^a D^{2n+1})^2 + \epsilon^{n+a} (D^{n+a})^2\} + (D^{2n+1})^2 = 0.$$

Proof. By using Proposition 2.1 we deduce

$$(7) \qquad \Delta = \det \left| \frac{\partial h^A}{\partial u^\alpha} \tilde{g}_{AB} \frac{\partial h^B}{\partial u^\beta} \right| = \sum_{A,B=1}^{2n+1} D^A M_{AB} D^B.$$

Next, using (6) in (7), we derive

$$\Delta = \sum_{a=1}^{2n} (D^a)^2 M_{aa} + 2 \sum_{a=1}^{n} D^a M_{a,2n+1} D^{2n+1} + (D^{2n+1})^2 M_{2n+1,2n+1}$$

$$= \sum_{a=1}^{2n} \epsilon^a (D^a)^2 + 2 \sum_{a=1}^{n} D^a D^{2n+1} \epsilon^a (-1)^a y^a + (D^{2n+1})^2 (1 + \sum_{a=1}^{n} \epsilon^a (y^a)^2)$$

(8)
$$= \sum_{a=1}^{n} (\epsilon)^a (D^a + (-1)^a y^a D^{2n+1})^2 + (D^{2n+1})^2 + \sum_{a=1}^{n} \epsilon^{n+a} (D^{n+a})^2.$$

Thus, our assertion follows from (5) and (8).

Proposition 2.2. *Let M be a degenerate hypersurface of R_s^{2n+1} given by (4). Then M is tangent to the structure vector field ξ if and only if $D^{2n+1} = 0$.*

Proof. By using (2) we see that ξ is tangent to M if and only if there exists $(X^1, \ldots, X^{2n}) \neq (0, \ldots 0)$ such that

$$\begin{cases} X^\alpha \frac{\partial h^a}{\partial u^\alpha} = 0, \\ X^\alpha \frac{\partial h^{n+a}}{\partial u^\alpha} = 0, \\ X^\alpha \frac{\partial h^{2n+1}}{\partial u^\alpha} = 2\epsilon, \ 1 \leq a \leq n, \ 1 \leq \alpha \leq 2n. \end{cases}$$

By standard arguments of linear algebra it follows that the above system has no trivial solution if and only if $D^{2n+1} = 0$.

Combining Theorem 2.1 with Proposition 2.3 we deduce the following result

Theorem 2.2. *Let M be a hypersurface of a (ϵ)-Sasakian manifold R_s^{2n+1}. Then M is a degenerate hypersurface tangent to the structure vector field ξ if and only if*

$$D^{2n+1} = 0 \quad \text{and} \quad \sum_{a=1}^{n} (\epsilon^a (D^a)^2 + \epsilon^{n+a} (D^{n+a})^2) = 0.$$

Main Theorem. *All hypersurfaces of a (ϵ)-Sasakian manifold R_n^{2n+1}, tangent to the structure vector field ξ, are not degenerate.*

Proof. Suppose that there exist a degenerate hypersurface M of a (ϵ)-Sasakian manifold R_n^{2n+1} tangent to the structure vector field ξ. Then from Theorem 2.2 and (4) we obtain $D^A = 0$, $1 \leq A \leq 2n+1$. Hence $rank\ [\frac{\partial h^A}{\partial u^\alpha}] < 2n$, which is a contradiction.

Corollary 2.1. *There do not exist degenerate hypersurfaces tangent to ξ in R_n^{2n+1}.*

Now, a natural question is whether in R_s^{2n+1}, with $s \neq n$, there exist degenerate hypersurfaces tangent to ξ. The answer is in the affirmative and it is supported by the following example. Consider R_1^5 with the (ϵ)-Sasakian structure from section 1 and M as the hypersurface given by equation $y^2 = x^1 + h(x^2 + y^1)$, where h is a smooth function. Then it is easy to check that

$$N = \frac{\partial}{\partial x^1} - h'\frac{\partial}{\partial x^2} + h'\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2} - (y^1 + y^2 h')\frac{\partial}{\partial z},$$

is tangent to M and $g(X, N) = 0, \forall\, X \in \Gamma(TM)$. Moreover, ξ is tangent to M. Thus M is a degenerate hypersurface tangent to ξ in R_1^5.

REFERENCES

1. M.Barros and A.Romero, *Indefinite Kähler manifolds*, Math. Ann., **261**(1982), 55-62.

2. A. Bejancu and K. L. Duggal, *Real Hypersurfaces of indefinite Kähler manifolds*, Internat. J. Math. & Math. Sci., **16**(3)(1993), 545-556.

3. A. Bejancu and K. L. Duggal, *Degenerate hypersurfaces of semi-Riemannian Manifolds* Bul. Inst. Polit. Iasi **37**(41)(1991), 13-22.

4. A. Kostrikin, *Introduction a l'algebre* Editions Mir, Moscow, 1981.

5. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

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