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SOME APPLICATIONS OF THE LAPLACE TRANSFORM RATIO ORDER

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ABSTRACT. Recently the new Laplace transform ratio stochastic order has been identified and studied in some detail. In this paper we first note a simple useful property of the Laplace transform ratio order. We apply this property to families of non-negative random variables that have the generalized semigroup property. We thus obtain a Laplace transform ratio order comparison for a host of non-negative random variables. Examples and applications of the resulting Laplace transform ratio order comparisons, which are of interest in reliability theory, are then described.

1. INTRODUCTION

Recently Shaked and Wong [4] have identified the new Laplace transform ratio stochastic order and studied it in some detail. In this paper we first note in Section 2 a simple useful property of the Laplace transform ratio order. We apply this property to families of non-negative random variables that have the generalized semigroup property. We thus obtain a Laplace transform ratio order comparison for a host of non-negative random variables. Examples and applications of the resulting Laplace transform ratio order comparisons, which are of interest in reliability theory, are then described in Section 3.

In this paper "increasing" and "decreasing" mean, respectively, " nondecreasing" and "nonincreasing".

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2. DEFINITIONS AND SOME BASIC RESULTS

For any non-negative random variable Z, with distribution function F_Z, the Laplace-Stieltjes transform is given by

$$L_Z(s) = \int_0^\infty e^{-st} dF_Z(t), \quad s > 0.$$

Let X be a non-negative random variable with distribution function F_X and Laplace transfrom L_X, and let Y be another nonn-negative random variable with distribution function F_Y and Laplace transform L_Y. The random variable X is said to be smaller than Y in the Laplace transform ratio order (denoted by $X \le_{Lt-r} Y$) if

$$\frac{L_Y(s)}{L_X(s)} \quad \text{is decreasing in } s > 0.$$

Shaked and Wong [4] have studied the Laplace transform ratio order in some detail. In particular they have shown that it is stronger than the Laplace transform order, but that it is weaker than the likelihood ratio and the reverse hazard rate orders (for the exact definitions of these stochastic orders see, e.g., Shaked and Wong [4] or Shaked and Shanthikumar (1994)).

The following lemma gives a useful property of the Laplace transform ratio order.

Lemma 2.1. *Let X and Z be two independent non-negative random variables. Then*

$$X \le_{Lt-r} X + Z.$$

Proof. Denote the Laplace transforms of X and of $X + Z$ by L_X and L_{X+Z}, respectively. Then, for $s > 0$ we have that

$$\frac{L_{X+Z}(s)}{L_X(s)} = L_Z(s),$$

and this decreases in $s > 0$.

Note that under the conditions of Lemma 2.1 we also have that $X \leq_{st} X+Z$, where "$\leq_{st}$" denotes the usual stochastic order (see its exact definition, e.g., in Section 1.A of Shaked and Shanthikumar (1994)). However, Shaked and Wong [4] have shown that the orders $\leq_{st}$ and $\leq_{Lt-r}$ do not imply each other.

In this paper we will apply the Laplace transform ratio order to familes of random variables of the form $\{X(\theta), \theta \in \Theta\}$. This notation stands for a collection of random variables with distribution functions parametrized by θ. That is, these random variables are associated with a family $\{F_\theta, \theta \in \Theta\}$ of univariate distribution functions, where F_θ is the distribution function of $X(\theta)$. Note that we are concerned here only with the marginal distributions of the $X(\theta)$'s, even if in some applications $\{X(\theta), \theta \in \Theta\}$ may be a well-defined stochastic process. In all the applications below Θ will be an interval in $(-\infty, \infty)$ or in $\{\ldots, -1, 0, 1, \ldots\}$.

The family of random variables $\{X(\theta), \theta \in \Theta\}$ is said to have the *semigroup property* if for every $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$, such that $\theta_1 < \theta_2$, we have that $\theta_2 - \theta_1 \in \Theta$ and that there exist independent random variables Z_1 and Z_2 such that

$$X(\theta_2) =_{st} Z_1 + Z_2, \quad Z_1 =_{st} X(\theta_1), \quad \text{and} \quad Z_2 =_{st} X(\theta_2 - \theta_1),$$

where "$=_{st}$" denotes equality in law. The family of random variables $\{X(\theta), \theta \in \Theta\}$ is said to have the *generalized semigroup property* if for every $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$, such that $\theta_1 < \theta_2$, there exist independent random variables Z_1 and Z_2 such that

$$X(\theta_2) =_{st} Z_1 + Z_2 \quad \text{and} \quad Z_1 =_{st} X(\theta_1). \tag{2.1}$$

Some examples of families of random variables that have the generalized semigroup property can be found in Shaked, Shanthikumar, and Tong (1995). It should be pointed out that the above definition of the generalized semigroup property is slightly different than Definition 2.8 (ii) of Shaked, Shanthikumar, and Tong (1995). However, all the examples in Remark 2.9 of Shaked, Shanthikumar, and Tong (1995) satisfy the above definition. Note that if a family of random variables $\{X(\theta), \theta \in \Theta\}$

satisfies the semigroup property then it also satisfies the generalized semigroup property. Shaked, Shanthikumar, and Tong (1995) also introduced definitions of sub-semigroup and super-semigroup properties. Again, if a family of random variables satisfies the sub-semigroup or the super-semigroup property then it also satisfies the generalized semigroup property. The results below hold for families that have the generalized semigroup property, and therefore they hold also for families that have the semigroup, the sub-semigroup, or the super-semigroup properties.

Using Lemma 2.1 the following result can be easily proven; it will be the key for the applications that follow.

Theorem 2.2. *Let $\{X(\theta), \theta \in \Theta\}$ be a family of non-negative random variables that have the generalized semigroup property such that, for any $\theta_1 < \theta_2$, the random variable Z_2 of (2.1) is non-negative a.s. Then*

$$X(\theta_1) \leq_{Lt-r} X(\theta_2) \quad \text{for } \theta_1 < \theta_2.$$

Proof. Since $X(\theta_1) =_{st} Z_1$ and $X(\theta_2) =_{st} Z_1 + Z_2$, where Z_1 and Z_2 are independent and non-negative, the stated result follows from Lemma 2.1.

3. EXAMPLES AND APPLICATIONS

Example 3.1. Let $X(\theta)$ be a Poisson random variable with mean $\mu(\theta) > 0$ that is increasing in θ. Then $\{X(\theta), \theta \in \Theta\}$ has the generalized semigroup property and it satisfies the conditions of Theorem 2.2. Therefore

$$X(\theta_1) \leq_{Lt-t} X(\theta_2) \quad \text{for } \theta_1 < \theta_2.$$

If $Y(\theta) = X(\theta) + 1$, where $X(\theta)$ is as above, then also $\{Y(\theta), \theta \in \Theta\}$ has the generalized semigroup property, and by Theorem 2.2 we have that

$$Y(\theta_1) \leq_{Lt-r} Y(\theta_2) \quad \text{for } \theta_1 < \theta_2.$$

Example 3.2. Let $X(\theta)$ be a Gamma random variable with a fixed scale parameter and with shape parameter $\alpha(\theta) > 0$ that is increasing in

θ. Then $\{X(\theta), \theta \in \Theta\}$ has the generalized semigroup property and it satisfies the conditions of Theorem 2.2.

Example 3.3. For $\theta \in \{1, 2, \ldots\}$, and a fixed $p \in (0, 1)$, let $X(\theta)$ be a binomial random variable with probabilities

$$P\{X(\theta) = x\} = \binom{\theta}{x} p^x (1-p)^{\theta - x}, \quad x = 0, 1, \ldots, \theta.$$

Then $\{X(\theta), \theta \in \{1, 2, \ldots\}\}$ has the semigroup property and it satisfies the conditions of Theorem 2.2. If $Y(\theta) = X(\theta) + 1$, where $X(\theta)$ is as above, then $\{Y(\theta), \theta \in \{1, 2, \ldots\}\}$ has the generalized semigroup property, and it also satisfies the conditions of Theorem 2.2.

Example 3.4. For $\theta \in \{1, 2, \ldots\}$, and a fixed $p \in (0, 1)$, let $X(\theta)$ be a negative binomial random variable with probabilities

$$P\{X(\theta) = x\} = \binom{x-1}{\theta-1} p^\theta (1-p)^{x-\theta}, \quad x = \theta, \theta + 1, \ldots.$$

Then $\{X(\theta), \theta \in \{1, 2, \ldots\}$ has the semigroup property and it satisfies the conditions of Theorem 2.2. If $Y(\theta) = X(\theta) + 1$, where $X(\theta)$ is as above, then $\{Y(\theta), \theta \in \{1, 2, \ldots\}\}$ has the generalized semigroup property, and it also satisfies the conditions of Theorem 2.2.

Application 3.5 (First passage times). Let $\{X(t), t \geq 0\}$ be a continuous-time strong Markov process with state space Θ, where Θ is an interval of the real line. Suppose that $\{X(t), t \geq 0\}$ has stationary transition probabilities. For $\theta_1, \theta_2 \in \Theta$, such that $\theta_1 < \theta_2$, let T_{θ_1, θ_2} denote the first passage time of the process to θ_2 given that $X(0) = \theta_1$. If, with probability 1, the sample paths of $\{X(t), t \geq 0\}$ have no positive jumps, then

$$T_{\theta_1,\theta_2} \leq_{Lt-r} T_{\theta_1,\theta_3}, \quad \theta_1 \leq \theta_2 \leq \theta_3, \tag{3.1}$$

and

$$T_{\theta_2,\theta_3} \leq_{Lt-r} T_{\theta_1,\theta_3}, \quad \theta_1 \leq \theta_2 \leq \theta_3. \tag{3.2}$$

These stochastic inequalities follow from Theorem 2.2 and from the easily verified facts that for each $\theta_1 \in \Theta$ we have that $\{T_{\theta_1,\theta}, \theta \in \Theta \cap (\theta_1, \infty)\}$

have the generalized semigroup property; and that for each $\theta_3 \in \Theta$ we have that $\{T_{-\theta,\theta_3}, \theta \in (-\Theta) \cap (-\theta_3, \infty)\}$ have the generalized semigroup property. Some processes of this kind are mentioned in Section 5 of Marshall and Shaked (1983).

In a similar fashion it can be shown that if the state space of the strong Markov process $\{X(t), t \in \mathcal{T}\}$ (where $\mathcal{T} = (0, \infty)$ or $\mathcal{T} = (0, 1, 2, \ldots\})$ is $\{0, 1, 2, \ldots\}$, and if the process is free of positive skips (that is, its sample paths cannot have positive jumps greater than 1), then (3.1) and (3.2) still hold. In particular, if $\mathcal{T} = \{0, 1, 2, \ldots\}$, and if N_{θ_1,θ_2} denote the first passage time of the process to θ_2 given that $X(0) = \theta_1$, and if the process is free of positive skips, then

$$N_{\theta_1,\theta_2} \leq_{Lt-r} N_{\theta_1,\theta_3}, \qquad \theta_1 \leq \theta_2 \leq \theta_3, \tag{3.3}$$

and

$$N_{\theta_2,\theta_3} \leq_{Lt-r} N_{\theta_1,\theta_3}, \qquad \theta_1 \leq \theta_2 \leq \theta_3. \tag{3.4}$$

Application 3.6 (Random minimums and maximums). Let X_1, $X_2, \ldots$ be a sequence of non-negative independent and identifically distributed random variables, and let N_1 be a positive integer-valued random variable that is independent of the X_i's. Denote $X_{(1:N_1)} \equiv \min\{X_1, X_2, \ldots, X_{N_1}\}$ and $X_{(N_1:N_1)} \equiv \max\{X_1, X_2, \ldots, X_{N_1}\}$. Let N_2 be another positive integer-valued random variable which is also independent of the X_i's and let $X_{(1:N_2)} \equiv \min\{X_1, X_2, \ldots, X_{N_2}\}$ and $X_{(N_2:N_2)} \equiv \max\{X_1, X_2, \ldots, X_{N_2}\}$. Shaked and Wong [4,5] have studied the properties of such random minimums and maximums. Among other things they obtained the following results:

$$N_1 \leq_{Lt-r} N_2 \Rightarrow X_{(1:N_2)} \leq_{hr} X_{(1:N_1)} \tag{3.5}$$

(that is, $\frac{P\{X_{(1:N_1)}>s\}}{P\{X_{(1:N_2)}>s\}}$ is increasing in $s \geq 0$; see Section 1.B in Shaked and Shanthikumar (1994)) and

$$N_1 \leq_{Lt-r} N_2 \Rightarrow X_{(N_1:N_1)} \leq_{rh} X_{(N_2:N_2)} \tag{3.6}$$

(that is, $\frac{P\{X_{(N_2:N_1)}\leq s\}}{P\{X_{(N_1:N_1)}\leq s\}}$ is increasing in $s \geq 0$; see Section 1.B in Shaked and Shanthikumar (1994)) . Some applications of these implications are described in Shaked and Wong [4,5]. Often it may not be easy to verify that $N_1 \leq_{Lt-r} N_2$. Theorem 2.2, through, e.g., Examples 3.1, 3.3, and 3.4, provides some instances in which the condition in (3.5) and in (3.6) holds.

Another instance in which the sufficient condition $N_1 \leq_{Lt-r} N_2$ in (3.5) and in (3.6) holds is provided by Application 3.5. Suppose that a production line produces one item in each production cycle. The number of the production cycles of that production line can often be modeled as the number of time units until a discrete-time strong Markov process, which starts at state 0, hits a certain threshold. Suppose that the items produced by the production line have independent and identically distributed lifetimes, and are put in parallel in some system. Then the lifetime of the system (using the notation in (3.3)) is $X_{(N_{0,\theta_2}:N_{0,\theta_2})}$, where θ_2 is the threshold mentioned above. If the production line is improved so that it produces items until it hits the threshold θ_3 (where $\theta_3 > \theta_2$), then the resulting system has the lifetime $X_{(N_{0,\theta_3}:N_{0,\theta_3})}$. By (3.3) we have that $N_{0,\theta_2} \leq_{Lt-r} N_{0,\theta_3}$, and therefore, by (3.6) it is seen that the improvement of the production line improves the lifetime of the resulting parallel system in the sense of the reverse hazard rate order. (The reverse hazard rate order is stronger than the usual stochastic order, and it yields sharper stochastic inequalities than the usual stochastic order; see, e.g., Chapter 1 in Shaked and Shanthikumar (1994).)

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