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REMARKS ON CERTAIN SALEM NUMBERS

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ABSTRACT. In this paper, we are interested in the localization in the real line of some Salem numbers obtained from Salem's construction. We introduce also a map from Salem numbers to Pisot numbers.

1. INTRODUCTION

A real algebraic integer $\theta > 1$ is said to be a Pisot (or Pisot-Vijayaraghavan) number if all its conjugates over the rational numbers field Q lie strictly within the unit circle. The set of such numbers S is known to be closed in the real line [7], and has a known minimum $\theta_0 = 1.3247...$ where $\theta_0^3 - \theta_0 - 1 = 0$ [8]. There is also an algorithm due essentially to J. Dufresnoy and C. Pisot [5] but developed by D. W. Boyd [1], [2], [3], to determine the structure of the set S in a finite interval. The set T of Salem numbers consists of those real algebraic integers $\tau > 1$ for which all other conjugates over Q lie within or on the unit circle, with at least one conjugate on the circle. By a construction due to Salem [7], every element of S is a limit of a sequence of elements of T. The questions whether the set $S \cup T$ is closed and whether $\inf T > 1$ are still unanswered.

In the first part of this expository paper we are interested in the polynomials $P_n(z) = z^n - 1$ which transform a given Salem number into Pisot numbers. This contrasts with the work of J. H. Silverman [9], who investigated the polynomials $P_n(z) = z^n - 1$ which transform a given Salem number into units.

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In the following parts we are interested in the localization in the real line of some Salem numbers obtained from Salem's construction. Mainly, we prove that there are arbitrarily large Salem numbers τ such that $\tau - 1$ is a unit.

2. A MAP FROM T TO S

From the definition of the set T, it follows that a Salem number τ has only one conjugate $\frac{1}{\tau}$ inside the unit circle and that its degree d is even.

Let $\alpha_1 = e^{i\pi x_1}, \alpha_2 = e^{i\pi x_2}, ..., \alpha_s = e^{i\pi x_s}$ be the conjugates of τ of modulus one in the upper half plane. Then, $d = 2s + 2 \geq 4$ and $0 < x_k < 1$ for all $1 \leq k \leq s$.

Firstly we have

Theorem 1. *Let τ be a Salem number. Then there exist infinitely many natural numbers n such that $\tau^n - 1$ is a Pisot number.*

Proof. It is known [7], that the numbers 1, $x_1, x_2, ..., x_s$ are linearly independent over Q and consequently the sequence of vectors $(nx_1, nx_2, ..., nx_s)_n$, where n is a natural number, is uniformly distributed modulo one in $[0,1]^s$. It follows that there exist infinitely many numbers n such that

$$0 < nx_k - E(nx_k) < \frac{1}{6} \qquad \text{for all } 1 \leq k \leq s,$$

where E is the integer part function.

Furthermore, as the function $\left|e^{ix} - 1\right|^2 = 2(1 - \cos x)$ is increasing on $[0, \frac{\pi}{3}]$, we obtain from the last inequality :

$$\left|\alpha_k^{2n} - 1\right| = \left|e^{i2\pi n x_k} e^{-i2\pi E(nx_k)} - 1\right| = \left|e^{i2\pi(nx_k - E(nx_k))} - 1\right| < \left|e^{i\frac{\pi}{3}} - 1\right| = 1,$$

for all $1 \leq k \leq s$.

To complete the proof it suffices to note that the inequality $\left|\frac{1}{\tau^{2n}} - 1\right| < 1$ is always true.

Now a natural question arises : what is the smallest natural number n_τ such that $\tau^{n_\tau} - 1$ is a Pisot number ? As the powers of a Salem number are also Salem numbers, this question is relevant for small elements of T. The following theorem gives a lower and an upper bounds for n_τ :

Theorem 2. *Let τ be a Salem number of degree d. Then,* $(d-1)\frac{\ln 2}{2\ln\tau} < n_\tau < 3^d$.

Proof. Let $H = 6^s > 1$. From Kronecker's theorem [7], there exist a natural number $n < H$ and rational integers p_1, p_2, ..., p_s such that

$$|nx_k - p_k| < \frac{1}{H^{\frac{1}{s}}} = \frac{1}{6},$$

for all $1 \leq k \leq s$. Furthermore, as the function $\left|e^{ix} - 1\right|^2 = 2(1 - \cos x)$ is increasing (resp. decreasing) on $[0, \frac{\pi}{3}]$ (resp. on $[-\frac{\pi}{3}, 0]$), we obtain from the last inequality

$$\left|\alpha_k^{2n} - 1\right| = \left|e^{i2\pi n x_k} e^{-i2\pi p_k} - 1\right| = \left|e^{i2\pi(nx_k - p_k)} - 1\right| < \left|e^{i\frac{\pi}{3}} - 1\right| = 1,$$

for all $1 \leq k \leq s$. Then, from the inequality $\left|\frac{1}{\tau^{2n}} - 1\right| < 1$, we deduce $n_\tau \leq 2n < 2(6)^s < 3^d$.

To obtain the lower bound, we first prove that if $\tau^n - 1$ is an element of the set S, then the algebraic integer $\tau^n + \frac{1}{\tau^n} - 1$ is a Pisot number with degree $\frac{d}{2} = s + 1$ and positive conjugates (such a number is called a totally positive Pisot number). Let σ be an embedding of $Q(\tau)$ in C not equal to the identity. Either $\sigma(\tau) = \frac{1}{\tau}$, and then we have

$$\sigma(\tau^n + \frac{1}{\tau^n} - 1) = \tau^n + \frac{1}{\tau^n} - 1,$$

or $\sigma(\tau) = \alpha_k^{\mp 1}$ for some $1 \leq k \leq s$, in which case we have

$$\sigma(\tau^n + \frac{1}{\tau^n} - 1) = \alpha_k^n + \frac{1}{\alpha_k^n} - 1 = 1 - \left|\alpha_k^n - 1\right|^2.$$

Then, from [10, Lemma 2], we obtain the inequality

$$\tau^n + \frac{1}{\tau^n} - 1 > 2^s,$$

which yields the result.

3. ON THE LOCALIZATION OF THE EXCEPTIONAL SALEM NUMBERS

The main tool of this part will be the next form of Salem's construction :

Let F be the minimal polynomial of a Pisot number θ of degree r and $F^*(x) = x^r F(\frac{1}{x})$. If $\frac{1}{\theta}$ is not a conjugate of θ, then there exists a real number N such that for $n > N$, the roots of the polynomial $x^n F(x) \pm F^*(x)$, are all of modulus ≤ 1 except one τ_n which is a Salem number. Furthermore, $Lim \tau_n = \theta$. Recall that it also follows from the definition of the set T that a Salem number is a unit. In what follows a Salem number τ such that $\tau - 1$ is a unit will be called an exceptional Salem number. It is easy to verify that almost all known small Salem numbers [4], are exceptional (30 of the 39 listed are exceptional). The next result shows that exceptional Salem numbers are not necessarily "near" 1.

Theorem 3. *Let θ be a totally positive Pisot number. If $\theta - 1$ is not a unit, then θ is a limit of a sequence of exceptional Salem numbers.*

To prove this result we need the following corollary of Salem's construction:

Lemma 1. *Let θ be a Pisot number such that $\frac{1}{\theta}$ is not a conjugate of θ. If $\theta - 1$ is a unit, then θ is a limit of a sequence of exceptional Salem numbers.*

Proof. Let F be the minimal polynomial of a Pisot number θ of degree r. Then, for n large and $n - r \equiv 1 (\bmod 2)$, the polynomial

$$R_n(x) = \frac{x^n F(x) + F^*(x)}{x+1},$$

has only one root $\tau_n > 1$ which is a Salem number.

As the resultant of the minimal polynomial of τ_n and the polynomial $x - 1$ must divide the resultant $R_n(1) = F(1) = \pm 1$ of the polynomials $R_n(x)$ and

$x-1$, we have that $\tau_n - 1$ is a unit.

Note that the smallest Pisot number of degree $d \geq 3$, whose minimal polynomial is $x^d - x^{d-1} - x^{d-2} + x^2 - 1$ [6], satisfies the statement of Lemma 1 and is a limit of a sequence of exceptional Salem numbers.

Proof of theorem 3. Let $F(x) = (x-\theta)(x-\theta^{(2)})...(x-\theta^{(r)})$ be the minimal polynomial of the Pisot number θ whose conjugates are all positive. Then, for $|z| = 1$ we have

$$\left|(z-\theta)(z-\theta^{(2)})...(z-\theta^{(r)})\right| \geqslant (\theta-1)(1-\theta^{(2)})...(1-\theta^{(r)}) = -F(1) = |F(1)|,$$

and

$$|F(z)| > |F(1)| - 1.$$

As $|F(1)| \geqslant 2$, we obtain from Rouché's theorem that the roots of the polynomial

$$Q_n(x) = x^n F(x) + |F(1)| - 1$$

are all of modulus < 1 except one θ_n. Furthermore, as $Q_n(1) = -1$, we have that θ_n is a Pisot number and $\theta_n - 1$ is a unit. From Lemma 1, we deduce that θ_n is a limit of a sequence of exceptional Salem numbers. Finally the inequality

$$|F(\theta_n)| = \frac{|F(1)| - 1}{\theta_n^n} \leq \frac{|F(1)| - 1}{\theta_0^n},$$

shows that $lim \theta_n = \theta$ and the result follows.

Let now θ be a Pisot number of degree one. When $\theta \geq 3$ (resp. when $\theta = 2$) it follows from Theorem 3 (resp. from Lemma 1) that θ is a limit of a sequence of exceptional Salem numbers, and consequently we obtain

Corollary. *There are arbitrarily large exceptional Salem numbers.*

4. ON THE LOCALIZATION OF OTHER SALEM NUMBERS

We can ask the same question about the localization of Salem numbers τ such that $\tau+1$ is a unit as almost all small Salem numbers [4] verify this property. The next results lead to the same conclusion as for exceptional Salem numbers.

Theorem 4. *There are arbitrarily large Salem numbers τ such that $\tau+1$ is a unit.*

To prove this theorem we need the next two lemmas.

Lemma 2. *Let θ be a Pisot number such that $\frac{1}{\theta}$ is not a conjugate of θ. If $\theta+1$ is a unit, then θ is a limit of a sequence of Salem numbers τ_n such that τ_n+1 is a unit.*

Proof. The proof is identical to the proof of Lemma 1 with $R_n(x)=\frac{x^nF(x)-F^*(x)}{x-1}$.

Note also that the smallest Pisot number of degree $d\geq 3$ satisfies the statement of Lemma 2.

Lemma 3. *Let θ be a Pisot number whose conjugates are all negative with at least one $<-\frac{1}{2}$. If the Pisot number $\theta+1$ is not a unit, then there exists a sequence of Pisot numbers θ_n such that θ_n+1 is a unit and $\lim\theta_n=\theta$.*

Proof. Let $F(x)=(x-\theta)(x-\theta^{(2)})...(x-\theta^{(r)})$ be the minimal polynomial of the Pisot number θ and z a complex number of modulus 1. From the inequalities

$$\left|(z-\theta)(z-\theta^{(2)})...(z-\theta^{(r)})\right|>(\theta-1)(1+\theta^{(2)})...(1+\theta^{(r)})=|F(-1)|\,\frac{\theta-1}{\theta+1}$$

and

$$\frac{1+\theta}{2}>|F(-1)|=(1+\theta)(1+\theta^{(2)})...(1+\theta^{(r)}),$$

we have

$$|F(z)|>|F(-1)|-1.$$

By Rouché's theorem, we have that the polynomial

$$x^nF(x)-|F(-1)|+1$$

has all its roots in the unit disç except one θ_n which is a Pisot number. Furthermore for $n \equiv r(\bmod 2)$, we obtain that $\theta_n + 1$ is a unit. Finally the inequality

$$|F(\theta_n)| = \frac{|F(-1)| - 1}{\theta_n^n} \leq \frac{|F(-1)| - 1}{\theta_0^n},$$

shows that $\lim \theta_n = \theta$.

Proof of theorem 4. Let θ_k be the Pisot number whose minimal polynomial is $z^2 - (3k - 2)z + (1 - 2k)$, where k is a natural number ≥ 2. An easy computation shows that θ_k satisfies the statement of Lemma 3. The result follows from Lemma 2.

It is clear now that there are arbitrarily large Salem numbers τ such that $\tau + 1$ or $\tau - 1$ is a unit.This does not happen, however, for Salem numbers τ such that $\tau - k$ is a unit, where k is a natural number ≥ 2.

Theorem 5.

(i) Let $k \geq 2$ be a natural number, then there exist infinitely many Salem numbers τ such that $\tau - k$ is a unit.

(ii) Let $k \geq 2$ be a natural number and τ be a Salem number of degree d. If $\tau - k$ is a unit, then $|\tau - k| < \frac{1}{\sqrt{2^d - 1}}$.

Proof.

(i) Let k be a natural number $\geqslant 2$. For n large enough, the polynomial

$$R_n(x) = \frac{x^{2n+1}(x - k) - (1 - kx)}{x^2 - 1},$$

has a root τ_n which is a Salem number. As $R_n(k) = 1$, the absolute value of the resultant of the polynomial $x - k$ and the minimal polynomial of τ_n is equal to 1. Consequently, $\tau_n - k$ is a unit and the theorem follows from the equality $lim \tau_n = k$.

(ii) Let $\alpha_1, \alpha_2, ..., \alpha_{\frac{d}{2}-1}$ be the conjugates of τ of modulus one in the upper half plane. If $\tau - k$ is a unit, then

$$|\tau - k|\,(k-1)^{d-1} < |\tau - k|\left|\frac{1}{\tau} - k\right| \prod_{1\leq i\leq d-2} |\alpha_i - k| = 1$$

and

$$|\tau - k| < \frac{1}{(k-1)^{d-1}}.$$

The result follows immediately from the last inequality when $k \geq 3$. For the case $k = 2$ we consider the function

$$f(x) = \frac{x-1}{x-2}.$$

From the equalities $|z-1|^2 = 2 - (z + \frac{1}{z})$ and $|z-2|^2 = 5 - 2(z+\frac{1}{z})$, where $|z| = 1$, we have

$$\left|\frac{z-1}{z-2}\right|^2 < \frac{2-(z+\frac{1}{z})}{4-2(z+\frac{1}{z})} = \frac{1}{2}$$

and

$$|f(z)| < \frac{1}{\sqrt{2}}.$$

The last inequality, together with

$$f([0,1]) = [0, \frac{1}{2}],$$

yields

$$1 \leq |f(\tau)|\left|f(\frac{1}{\tau})\right| \prod_{1\leq i\leq d-2} |f(\alpha_i)| < \frac{|f(\tau)|}{\sqrt{2^d}},$$

and

$$-\frac{1}{\sqrt{2^d}-1} < -\frac{1}{\sqrt{2^d}+1} < \tau - 2 < \frac{1}{\sqrt{2^d}-1}.$$

This completes the proof of the theorem 5 (ii).

Remarks.

1. From theorem 5 (ii), we deduce that the integer part of τ is k.

2. With the notations of the proof of the theorem 5 (ii) and from the inequalities

$$|\tau + k|\left|\frac{1}{\tau} + k\right| \prod_{1 \leq i \leq d-2} |\alpha_i + k| \geqslant (\tau + 2)(\frac{1}{\tau} + 2) > 9,$$

we obtain that there is no Salem number τ such that $\tau + k$ is a unit.

3. Salem's construction is not sufficient to consider all linear polynomials. For example we don not know if there exists a natural number $k \geq 2$ for which there is infinitely many Salem numbers τ such $k\tau - (k+1)$ is a unit (or with a bounded norm).

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