Arab J. Math. Sc.
Volume 15, Number 2, December 2009
pp 1-11

Integrability of det∇u and evolutionary Wente's problem associated to reaction-diffusion operator

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ABSTRACT. In this paper, we consider the solution of evolutionary Wente's problem with the wavefronts operator for a global reaction-diffusion population model in $\mathbb{R}^+ \times \mathbb{R}^2$. We study in particular the best constant in the so-called Wente's inequality. We consider the best constant associated to the L^∞ norm of this solution.

Keywords. Jacobian determinant, Wente's inequality, Best constant.
Mathematics Subject Classification (2000). 35K55, 35K05.

1. INTRODUCTION

The classical Wente's problem arises in the study of constant mean curvature immersions (see [9]), for which the scalar version of equation is just the following problem:

$$
\begin{cases} -\Delta\psi & = \det\nabla v = a_{x_1}b_{x_2} - a_{x_2}b_{x_1} & \text{in } \Omega \\ \psi & = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}
$$

where $x = (x_1, x_2)$, a, b are functions defined in Ω. If $\Omega = \mathbb{R}^2$, we shall replace the boundary condition by the ground state condition

$$
\lim_{|x|\to+\infty} \psi(x) = 0,
$$

where $|x|$ is the Euclidean norm $|x| = (x_1^2 + x_2^2)^{1/2}$. In both case, when $v = (a, b) \in H^1(\Omega, \mathbb{R}^2)$, it is proved in [10, 4] that ψ, the solution of (1.1)

Supported by College of Sciences (KSU) Research Center project No. Math/2010/31.
Date received. December 01, 2009

exists, lies in $C(\bar{\Omega})$ and $\nabla\psi$ in $L^2(\Omega)$. More precisely, we have

$$\|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}. \tag{1.2}$$

Many works have been done to estimate the best constant, see for example [1, 8, 7] and some generalizations in [3, 5].

Here, we deal with the following problem: Let $u \in H^1_{loc}\left(\mathbb{R}^+, H^1(\mathbb{R}^2, \mathbb{R}^2)\right)$, then we consider the equation

$$\left\{\begin{array}{rll} \partial_t\varphi(t,x) - D\Delta_x\varphi(t,x) + \gamma\varphi(t,x) & = \det\nabla u(t,x) & \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^2 \\ \lim\limits_{|x|\to+\infty} \varphi(t,x) & = 0 & \forall \ t > 0 \\ \varphi(0,x) & = 0 & \text{in} \quad \mathbb{R}^2, \end{array}\right. \tag{1.3}$$

where D and γ are positives constants which represent the diffusion and death rate of mature population. Here $H^1(I, E)$ denotes the standard Sobolev space of functions in $L^2(I)$ such that the derivative is also in $L^2(I)$ where I is an interval in $\mathbb{R}$ and E is a Banach space.

It is not trivial that a solution exists for (1.3), since the second member lies apparently just in $L^1(\mathbb{R}^2)$. But we know that $\det\nabla u$ has a special structure which admits some higher integrability than L^1, it lies indeed in the Hardy space $\mathcal{H}^1$ (see [6]). Here we will use the special form of determinant to show that a global solution φ exists and moreover $\|\varphi\|_{L^\infty(\mathbb{R}^2)}$ is locally bounded. We can get nearly the best estimate for it's L^∞ norm.
Given $\gamma = 0$ and $D = 1$, then problem (1.3), is simply given in [2], by

$$\left\{\begin{array}{rll} \partial_t\varphi(t,x) - \Delta_x\varphi(t,x) & = \det\nabla u(t,x) & \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^2 \\ \lim\limits_{|x|\to+\infty} \varphi(t,x) & = 0 & \forall \ t > 0 \\ \varphi(0,x) & = 0 & \text{in} \quad \mathbb{R}^2. \end{array}\right. \tag{1.4}$$

The authors in [2] derived a representation formula for the solutions of (1.4) which are defined in $C\left(\mathbb{R}^+ \times \mathbb{R}^2\right)$ and the best constant which appears in the corresponding inequality is equals to $(2\pi)^{-1}$.
First, thanks to the linearity of our problem, we can decompose the solution φ as $\varphi_1 + \varphi_2$, where

$$\left\{\begin{array}{rll} \partial_t\varphi_2(t,x) - D\Delta_x\varphi_2(t,x) + \gamma\varphi_2(t,x) & = 0 & \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^2 \\ \lim\limits_{|x|\to+\infty} \varphi_2(t,x) & = 0 & \forall \ t > 0 \\ \varphi_2(0,x) & = -\varphi_0(x) & \text{in} \quad \mathbb{R}^2 \end{array}\right. \tag{1.5}$$

and

(1.6)

$$\begin{cases} \partial_t\varphi_1(t,x) - D\Delta_x\varphi_1(t,x) + \gamma\varphi_1(t,x) &= \det\nabla u(t,x) \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^2 \\ \lim\limits_{|x|\to+\infty} \varphi_1(t,x) &= 0 \qquad\qquad\qquad \forall\ t>0 \\ \varphi_1(0,x) &= \varphi_0(x) \qquad\qquad\quad \text{in} \quad \mathbb{R}^2, \end{cases}$$

where φ_0 is the solution of classical Wente's problem in $\mathbb{R}^2$, associated to $u(0,x)$:

$$\text{(1.7)} \qquad \begin{cases} -D\Delta\varphi_0(x) &= \det\nabla u(0,x) \quad \text{on} \quad \mathbb{R}^2 \\ \lim\limits_{|x|\to+\infty} \varphi_0(x) &= 0. \end{cases}$$

It is well-known that $\varphi_2(t,x)$ is given by $\varphi_2 = -E(t,\cdot) * \varphi_0(x)$, where

$$E(t,x) = \frac{1}{4D\pi t} e^{-\frac{|x|^2}{4Dt} - \gamma t}$$

denotes the fundamental solution of heat operator in $\mathbb{R}^2$. In other words, E satisfies

$$\partial_t E - D\Delta_x + \gamma E = \delta_{(0,0)}.$$

By the limit condition for φ_0, it is easy to see that $\|\varphi_2\|_\infty \leq \|\varphi_0\|_\infty$ and

$$\lim_{t\to\infty} \|\varphi_2\|_\infty = 0.$$

Thus our study will concentrate on that of φ_1.

Throughout this paper, $\|\cdot\|_p$ denotes always the L^p norm over $\mathbb{R}^2$, ∇ and Δ denote always the derivation to the variable x. In the following we denote by

$$\text{(1.8)} \qquad \phi(t,\cdot) = e^{\gamma t}\varphi_1(t,.) - \varphi_0$$

and

$$G_{u,\gamma}(t) = \int_0^t e^{\gamma s}\big[\|\nabla(\partial_s a)(s,.)\|_2\|\nabla b(s,\cdot)\|_2 + \|\nabla a(s,\cdot)\|_2\|\nabla(\partial_s b)(s,\cdot)\|_2 + \gamma\|\nabla a(s,\cdot)\|_2\|\nabla b(s,\cdot)\|_2\big]\,ds.$$

Note that $u \in H^1_{loc}\left(\mathbb{R}^+, H^1(\mathbb{R}^2,\mathbb{R}^2)\right)$ implies that $G_{u,\gamma}(t) < \infty$ for any t. We define also $\Sigma_\gamma(u) = G_{u,\gamma}(\infty)$. We have the following result.

Theorem 1.1. *Let u be a function in $H^1_{loc}\left(\mathbb{R}^+, H^1(\mathbb{R}^2,\mathbb{R}^2)\right)$, then a unique global solution of (1.6) exists and $\varphi_1 \in C\left(\mathbb{R}^+ \times \mathbb{R}^2\right)$. Furthermore,*

$$\sup_{\Sigma_\gamma(u)\neq 0}\sup_{t>0}\frac{\|\phi(t,\cdot)\|_\infty}{G_{u,\gamma}(t)} = \frac{1}{2\pi D}, \tag{1.9}$$

where a and b are the two components of u, i.e. $u(t,x) = (a,b)(t,x)$.

Remark 1.1. Consequently, we get $\varphi \in C\left(\mathbb{R}^+ \times \mathbb{R}^2\right)$.

Theorem 1.2. *The solution φ_1 of (1.6) belongs to $C\left(\mathbb{R}^+, H^1(\mathbb{R}^2)\right)$ if $u \in H^1_{loc}\left(\mathbb{R}^+, H^1(\mathbb{R}^2,\mathbb{R}^2)\right)$. Furthermore, $t \mapsto \varphi_1(t,\cdot)$ is locally Lipschitz in $L^2(\mathbb{R}^2)$ and we have the following estimates: For any $t > 0$,*

$$\frac{1}{2}\|\phi(t,\cdot)\|_2^2 + D\int_0^t \|\nabla\phi(s,\cdot)\|_2^2 ds \le \frac{1}{4\pi D}\int_0^t G^2_{u,\gamma}(s)ds \tag{1.10}$$

and

$$\|\partial_t\phi\|^2_{L^2([0,t]\times\mathbb{R}^2)} + \frac{D}{2}\|\nabla\phi(t,.)\|_2^2 \le \frac{3}{4\pi D}G^2_{u,\gamma}(t). \tag{1.11}$$

2. Proof of Theorem 1

When $u \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2)$, we know that the solution of (1.6) is explicitly given by:

$$\varphi_1(t,x) = E(t,.) * \varphi_0(x) + \int_0^t E(s,\cdot) * \det(\nabla u)(t-s,.)(x)ds. \tag{2.1}$$

We will establish the estimate (1.9) in this case, then the existence and estimate of φ_1 in general case will come from density arguments. First, we consider the value of φ_1 at the point $(t,0)$, we have

(2.2)

$$\varphi_1(t,0) = \int_{\mathbb{R}^2}\frac{e^{-\frac{|y|^2}{4Dt}-\gamma t}}{4D\pi t}\varphi_0(y)dy + \int_0^t\int_{\mathbb{R}^2}\frac{e^{-\frac{|y|^2}{4Ds}-\gamma s}}{4D\pi s}\det(\nabla u)(t-s,y)dyds = I + J.$$

Using polar coordinates, we get

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^{+\infty}\frac{e^{-\frac{r^2}{4Dt}-\gamma t}}{4D\pi t}\varphi_0(r,\theta)rdrd\theta \\ &= -\frac{e^{-\gamma t}}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\partial_r\left(e^{-\frac{r^2}{4Dt}}\right)\varphi_0(r,\theta)drd\theta \\ &= \frac{e^{-\gamma t}}{2\pi}\int_0^{2\pi}\int_0^{+\infty} e^{-\frac{r^2}{4Dt}}\partial_r\varphi_0 drd\theta + e^{-\gamma t}\varphi_0(0). \end{aligned}$$

Since

$$\det\nabla u = \frac{(a_r b)_\theta - (a_\theta b)_r}{r},$$

we have

$$J = \int_0^t \int_0^{2\pi} \int_0^{+\infty} \frac{e^{-\frac{r^2}{4Ds}-\gamma s}}{4D\pi s} \Big[(a_r b)_\theta - (a_\theta b)_r\Big](t-s,r,\theta) dr d\theta ds$$

$$= \int_0^t \int_0^{2\pi} \int_0^{+\infty} \partial_r \left(\frac{e^{-\frac{r^2}{4Ds}-\gamma s}}{4D\pi s} \right) (a_\theta b)(t-s,r,\theta) dr d\theta ds.$$

It is easy to see that

$$\partial_r \left(\frac{e^{-\frac{r^2}{4Ds}-\gamma s}}{4D\pi s} \right) = \partial_s \left(\frac{e^{-\frac{r^2}{4Ds}-\gamma s}}{2D\pi s} \right) - \frac{\gamma e^{-\frac{r^2}{4Ds}-\gamma s}}{2D\pi r}.$$

Then

$$J = -\int_0^t \int_0^{2\pi} \int_0^{+\infty} \partial_s \left(\frac{e^{-\frac{r^2}{4Ds}-\gamma s}}{2D\pi r} \right) (a_\theta b)(t-s,r,\theta) dr d\theta ds$$

$$- \int_0^t \int_0^{2\pi} \int_0^{+\infty} \frac{\gamma e^{-\frac{r^2}{4Ds}-\gamma s}}{2D\pi r} (a_\theta b)(t-s,r,\theta) dr d\theta ds.$$

Thus

$$\varphi_1(t,0) - e^{-\gamma t}\varphi_0(0)$$

$$= e^{-\gamma t} \int_0^{2\pi} \int_0^{+\infty} \frac{e^{-\frac{r^2}{4Dt}}}{2\pi} \partial_r \varphi_0 dr d\theta - e^{-\gamma t} \int_0^{+\infty} \int_0^{2\pi} \frac{e^{-\frac{r^2}{4Dt}}}{2D\pi r} (a_\theta b)(0,r,\theta) dr d\theta$$

$$- \int_0^t \int_0^{2\pi} \int_0^{+\infty} \frac{e^{-\frac{r^2}{4Ds}-\gamma s}}{2D\pi r} \partial_s (a_\theta b)(t-s,r,\theta) dr d\theta ds$$

$$- \int_0^t \int_0^{2\pi} \int_0^{+\infty} \frac{\gamma e^{-\frac{r^2}{4Ds}-\gamma s}}{2D\pi r} (a_\theta b)(t-s,r,\theta) dr d\theta ds.$$

On the other hand, $-D\Delta\varphi_0 = \det\nabla u(0,x)$ means

$$-\frac{1}{r}\partial_r (r\partial_r \varphi_0) - \frac{1}{r^2}\partial_\theta^2 \varphi_0 = \frac{(a_r b)_\theta - (a_\theta b)_r}{Dr},$$

so

$$-\partial_r \left[r\partial_r \varphi_0 - \frac{a_\theta b}{D} \right] = \partial_\theta \left(\frac{\partial_\theta \varphi_0}{r} + \frac{a_r b}{D} \right).$$

Therefore

$$Dr\partial_r\varphi_0(r,\theta)-(a_\theta b)(0,r,\theta)=-\int_0^r\partial_\theta\left[\frac{1}{\sigma}(\partial_\theta\varphi_0)(\sigma,\theta)+(a_rb)(0,\sigma,\theta)\right]d\sigma.$$

Then,

$$\varphi_1(t,0)-e^{-\gamma t}\varphi_0(0)=\int_0^{+\infty}\int_0^{2\pi}\frac{e^{-\frac{r^2}{4t}-\gamma t}}{2D\pi r}\left[Dr\partial_r\varphi_0(r,\theta)-a_\theta b(0,r,\theta)\right]drd\theta$$

$$-\int_0^t\int_0^{2\pi}\int_0^{+\infty}\frac{e^{-\frac{r^2}{4Ds}-\gamma s}}{2D\pi r}\partial_s(a_\theta b)(t-s,r,\theta)drd\theta ds$$

$$-\int_0^t\int_0^{2\pi}\int_0^{+\infty}\frac{\gamma e^{-\frac{r^2}{4Ds}-\gamma s}}{2D\pi r}(a_\theta b)(t-s,r,\theta)drd\theta ds$$

$$=-\int_0^t\int_0^{2\pi}\int_0^{+\infty}\frac{\gamma e^{-\frac{r^2}{4D(t-s)}-\gamma(t-s)}}{2D\pi r}(a_\theta b)(s,r,\theta)drd\theta ds$$

$$-\int_0^t\int_0^{+\infty}\int_0^{2\pi}\frac{e^{-\frac{r^2}{4D(t-s)}-\gamma(t-s)}}{2D\pi r}\partial_s\left(a_\theta b\right)(s,r,\theta)drd\theta ds$$

$$=-\int_0^t\int_0^{+\infty}\int_0^{2\pi}\frac{e^{-\frac{r^2}{4D(t-s)}-\gamma t}}{2D\pi r}\partial_s\left(e^{\gamma s}(a_\theta b)\right)(s,r,\theta)drd\theta ds.$$

We get finally

$$\phi(t,0)=-\int_0^t\int_0^{+\infty}\int_0^{2\pi}\frac{e^{-\frac{r^2}{4D(t-s)}}}{2D\pi r}\partial_s\left(e^{\gamma s}(a_\theta b)\right)(s,r,\theta)drd\theta ds. \tag{2.3}$$

If we denote by

$$\bar{b}(s,r)=\frac{1}{2\pi}\int_0^{2\pi}b(s,r,\theta)d\theta$$

we have

$$\int_0^{2\pi}|b-\bar{b}|^2d\theta\le\int_0^{2\pi}b_\theta^2d\theta,\quad\forall\, b\in H^1(0,2\pi).$$

Then

$$
\begin{aligned}
|\phi(t,0)| &\leq \frac{1}{2\pi D}\int_0^t e^{\gamma s}\int_0^{+\infty}\frac{1}{r}\int_0^{2\pi}\left|\partial_s a_\theta\left[b-\bar{b}(s,r)\right]\right| + \left|a_\theta\partial_s\left[b-\bar{b}(s,r)\right]\right| d\theta dr ds\\
&+\frac{\gamma}{2\pi D}\int_0^t e^{\gamma s}\int_0^{+\infty}\frac{1}{r}\int_0^{2\pi}\left|a_\theta\left[b-\bar{b}(s,r)\right]\right| d\theta dr ds\\
&\leq \frac{1}{2\pi D}\int_0^t e^{\gamma s}\|\nabla(\partial_s a)(s,\cdot)\|_2\|\nabla b(s,\cdot)\|_2+\|\nabla a(s,\cdot)\|_2\|\nabla(\partial_s b)(s,\cdot)\|_2 ds\\
&+\frac{\gamma}{2\pi D}\int_0^t e^{\gamma s}\|\nabla(a)(s,\cdot)\|_2\|\nabla b(s,\cdot)\|_2 ds\\
&\leq \frac{1}{2\pi D}G_{u,\gamma}(t).
\end{aligned}
$$

The last inequality comes from

$$
\begin{aligned}
&\int_0^{+\infty}\frac{1}{r}\int_0^{2\pi}\left|\partial_s a_\theta\left[b-\bar{b}(s,r)\right]\right| dr d\theta\\
&\leq \int_0^{+\infty}\frac{1}{r}\|\partial_s a_\theta\|_{L^2(0,2\pi)}\|b-\bar{b}(s,r)\|_{L^2(0,2\pi)}dr\\
&\leq \left[\int_0^{+\infty}\int_0^{2\pi}\frac{(\partial_s a_\theta)^2}{r}d\theta dr\right]^{1/2}\left[\int_0^{+\infty}\int_0^{2\pi}\frac{(\partial_\theta b)^2}{r}d\theta dr\right]^{1/2}\\
&\leq \|\nabla\partial_s a(s,\cdot)\|_2\|\nabla b(s,\cdot)\|_2.
\end{aligned}
$$

Since equation (1.1) is invariant by translation with respect to the variable x, so we get the same estimate for all x by considering $\varphi_1(x+\cdot)$, hence

$$
|\phi(t,x)| \leq \frac{1}{2\pi D}G_{u,\gamma}(t), \quad \forall\, t\in\mathbb{R}^+, x\in\mathbb{R}^2. \tag{2.4}
$$

For the inverse inequality, let $h:\mathbb{R}^+\to\mathbb{R}^+$ be a smooth, decreasing, compactly supported function such that $h(0)=1$. Take now $u(s,x)=h(s)u_0(x)$ where u_0 will be determined later, and φ_0 the solution of classical Wente's problem corresponding to u_0. So the solution of associated heat equation (1.6) is explicitly given by (2.1). We will look at the value of $\varphi_1(t,0)$ given by (2.2).

If we take $u_0 = (a_0, b_0) = g(r)x$ with g a regular radial function with compact support in $\mathbb{R}^2$, such that $u_0 \in H^1(\mathbb{R}^2)$, then

$$\det(\nabla u)(s,x) = h^2(s)\det(\nabla u_0)(x) = \frac{h^2(s)}{2r}\left[r^2 g^2(r)\right]'.$$

According to (2.3), we obtain that

$$\begin{aligned}\phi(t,0) &= -\frac{1}{2\pi D}\int_0^\infty \int_0^t \int_0^{2\pi} \left[e^{\gamma s} h^2(s)\right]' e^{-\frac{r^2}{4D(t-s)}} r g^2(r)\cos^2\theta d\theta ds dr \\ &= -\frac{1}{2D}\int_0^\infty \int_0^t \left[e^{\gamma s} h^2(s)\right]' e^{-\frac{r^2}{4D(t-s)}} r g^2(r) ds dr.\end{aligned}$$

Let $h(s) = e^{-(1+\gamma)s/2}$, then clearly

$$\lim_{t\to\infty} |\phi(t,0)| = \frac{1}{2D}\int_0^\infty r g^2(r)dr = \varphi_0(0).$$

Otherwise, a direct calculus shows that

$$G_{u,\gamma}(t) = -2\pi \int_0^t e^{\gamma s/2} h(s)[e^{\gamma s/2} h(s)]' ds \times \int_0^\infty \sigma^3 g'^2(\sigma) d\sigma.$$

For t large enough, we get

$$G_{u,\gamma}(t) = \pi \int_0^\infty \sigma^3 g'^2(\sigma) d\sigma = \|\nabla a_0\|_2 \|\nabla b_0\|_2.$$

In conclusion, we find in this special case

$$\lim_{t\to\infty} \frac{|\phi(t,0)|}{G_{u,\gamma}(t)} = \frac{\varphi_0(0)}{\|\nabla a_0\|_2 \|\nabla b_0\|_2}.$$

Then

$$\begin{aligned}&\sup_{\Sigma_\gamma(u)\neq 0} \sup_{t>0} \frac{\|\phi(t,\cdot)\|_\infty}{G_{u,\gamma}(t)} \\ &\geq \sup_{\Sigma_\gamma(u)\neq 0} \lim_{t\to\infty} \frac{|\phi(t,\cdot)|}{G_{u,\gamma}(t)} \geq \frac{1}{2\pi D} \frac{\displaystyle\int_0^{+\infty} r g^2(r)dr}{\displaystyle\int_0^{+\infty} r^3 g'(r)^2 dr}.\end{aligned} \tag{2.5}$$

Choosing $g_\varepsilon(r) = r^{\varepsilon-1}e^{-r/2}$ with $\varepsilon > 0$, then by [1], $\lim_{\varepsilon\to 0} \dfrac{\int_0^{+\infty} r g_\varepsilon^2(r)dr}{\int_0^{+\infty} r^3 g_\varepsilon'(r)^2 dr} = 1$

and we deduce that

$$\sup_{\Sigma_\gamma(u)\neq 0} \sup_{t>0} \frac{\|\phi(t,\cdot)\|_\infty}{G_{u,\gamma}(t)} \geq \frac{1}{2\pi D}. \tag{2.6}$$

Finally, the proof is then done. □

3. Proof of Theorem 2

We recall that

$$\phi(t,x) = e^{\gamma t}\varphi_1(t,x) - \varphi_0(x).$$

First, we will prove (1.10). Thanks to density arguments, we can just consider the case where $u \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2)$. By equations (1.6) and (1.7), we have

$$\begin{aligned}
\partial_t\phi - D\Delta\phi + \gamma\phi &= e^{\gamma t}\det\nabla u(t,x) - \det\nabla u(0,x) + \gamma e^{\gamma t}\varphi_1(t,x) - \gamma\varphi_0(x) \\
&= \int_0^t \partial_\sigma\left[e^{\gamma\sigma}\det\nabla u\right](\sigma,x)d\sigma + \gamma\int_0^t \partial_\sigma\left[e^{\gamma\sigma}\varphi_1\right](\sigma,x)d\sigma \\
&= \int_0^t \partial_\sigma\left[e^{\gamma\sigma}\det\nabla u\right](\sigma,x)d\sigma + \gamma\int_0^t \partial_\sigma\phi(\sigma,x)d\sigma.
\end{aligned}$$

Now, we are just to concentrate our attention to the following equation

$$\partial_t\phi - D\Delta\phi = \int_0^t \partial_\sigma\left[e^{\gamma\sigma}\det\nabla u\right](\sigma,x)d\sigma. \tag{3.1}$$

Multiplying (3.1) by ϕ and integrating with respect to t, we get

$$\frac{1}{2}\phi(t,x)^2 - D\int_0^t \phi(s,x)\Delta\phi(s,x)ds = \int_0^t \phi(s,x)\int_0^s \partial_\sigma\left[e^{\gamma\sigma}\det\nabla u\right](\sigma,x)d\sigma ds.$$

Now we integrate over $\mathbb{R}^2$, then

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}\phi(t,x)^2dx + D\int_0^t\int_{\mathbb{R}^2}|\nabla\phi(s,x)|^2dxds \\
&= \int_{\mathbb{R}^2}\int_0^t \phi(s,x)\int_0^s \partial_\sigma\left[e^{\gamma\sigma}\det\nabla u\right](\sigma,x)d\sigma dsdx.
\end{aligned}$$

Hence, by (2.4), we get

$$\begin{aligned}&\frac{1}{2}\|\phi(t,\cdot)\|_2^2+D\int_0^t\|\nabla\phi(s,\cdot)\|_2^2ds\\&\le\int_0^t\int_0^s\|\phi(s,\cdot)\|_\infty\int_{\mathbb{R}^2}\Big|\partial_\sigma\left[e^{\gamma\sigma}\det\nabla u\right](\sigma,x)\Big|dxd\sigma ds\\&\le\frac{1}{2\pi D}\int_0^t\int_0^s G_{u,\gamma}(s)\int_{\mathbb{R}^2}\Big|\partial_\sigma\left[e^{\gamma\sigma}\det\nabla u\right](\sigma,x)\Big|dsd\sigma dx.\end{aligned}$$

Otherwise, we have

$$\partial_t\left[e^{\gamma t}\det\nabla u\right]=\gamma e^{\gamma t}\det\nabla u+e^{\gamma t}\det\left[\nabla(\partial_t a),\nabla b\right]+e^{\gamma t}\det\left[\nabla a,\nabla(\partial_t b)\right],$$

which implies

$$\begin{aligned}\int_{\mathbb{R}^2}\Big|\partial_t\left[e^{\gamma t}\det\nabla u\right](\sigma,x)\Big|dx\le\; &e^{\gamma t}\|\nabla(\partial_t a)(\sigma,\cdot)\|_2\|\nabla b(\sigma,\cdot)\|_2+\\ e^{\gamma t}\left(\|\nabla a(\sigma,\cdot)\|_2\|\nabla(\partial_t b)(\sigma,\cdot)\|_2+\gamma\|\nabla a(\sigma,\cdot)\|_2\|\nabla b(\sigma,\cdot)\|_2\right)&=G'_{u,\gamma}(\sigma).\end{aligned}\tag{3.2}$$

Finally we obtain

$$\begin{aligned}&\frac{1}{2}\|\phi(t,\cdot)\|_2^2+D\int_0^t\|\nabla\phi(s,\cdot)\|_2^2ds\\&\le\frac{1}{2\pi D}\int_0^t\int_0^s G_{u,\gamma}(s)G'_{u,\gamma}(\sigma)d\sigma ds=\frac{1}{4\pi D}\int_0^t G_{u,\gamma}(s)^2ds\end{aligned}\tag{3.3}$$

and (1.10) follows.

For proving (1.11), multiplying (3.1) by $\partial_t\phi$ and integrating with respect to t and x, we get easily

$$\begin{aligned}&\|\partial_t\phi\|^2_{L^2([0,t]\times\mathbb{R}^2)}+\frac{D}{2}\|\nabla\phi(t,.)\|_2^2\\&=-\int_{\mathbb{R}^2}\int_0^t\phi(s,x)\partial_s\left[e^{\gamma s}\det\nabla u\right](s,x)dsdx+\int_{\mathbb{R}^2}\phi(t,x)\int_0^t\partial_s\left[e^{\gamma s}\det\nabla u\right](s,x)dsdx.\end{aligned}$$

Then

$$\|\partial_t\phi\|^2_{L^2([0,t]\times\mathbb{R}^2)}+\frac{D}{2}\|\nabla\phi(t,.)\|_2^2\le I_1+J_1.$$

Using (2.4) and (3.2), we get

$$|I_1|\le\frac{1}{2\pi D}\int_0^t G_{u,\gamma}(s)G'_{u,\gamma}(s)ds=\frac{1}{4\pi D}G_{u,\gamma}(t)^2$$

and

$$|J_1|\le\frac{1}{2\pi D}G_{u,\gamma}(t)\int_0^t G'_{u,\gamma}(s)ds=\frac{1}{2\pi D}G_{u,\gamma}(t)^2.$$

The proof of (1.11) is then completed.

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