

DECAY AND GRADIENT ESTIMATE FOR SOLUTIONS OF A QUASILINEAR HEAT EQUATION

Salim A. Messaoudi

ABSTRACT. In this work we consider an initial boundary value problem related to the equation

$$u_t - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) = b|u|^{p-2}u, \quad p > \alpha \geq 2, \quad b > 0.$$

We give, under suitable conditions on the initial data, a precise estimate on the gradient and prove that the energy of weak solutions decay exponentially for $\alpha = 2$ and in a polynomial rate for $\alpha > 2$ as $t \rightarrow \infty$.

1. INTRODUCTION

Research of global existence and finite time blow up of solutions for the initial boundary value problem

$$(1.1) \quad \begin{aligned} u_t - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) + f(u) &= 0, & x \in \Omega, & \quad t > 0 \\ u(x, t) &= 0, & x \in \partial\Omega, & \quad t \geq 0 \\ u(x, 0) &= u_0(x), & x \in \Omega, & \end{aligned}$$

where $\alpha \geq 2$ and Ω is a bounded domain of \mathbb{R}^n ($n \geq 1$), with a smooth boundary $\partial\Omega$, has attracted a great deal of people. The obtained results show that global existence and nonexistence depend roughly on α , the degree of nonlinearity in f , the dimension n , and the size of the initial data. In the early 70's, Levine [6] introduced the concavity method and showed that solutions with negative energy blow up in finite time. Later, this method had been improved by Kalantarov and Ladyzhenskaya [5] to accommodate more situations. This type of results have been extensively generalized and improved by Levine, Park, and Serrin in [7]. The authors, in these papers,

Subject classification: 35K05 - 35K65.

proved several global and nonglobal existence theorems. On the other hand if f has at most a linear growth then we can find global solutions.

Concerning the asymptotic behavior, Engler, Kawohl, and Luckhaus [2] considered (1.1), with $\alpha = 2$ and showed that for, $f(0) = 0$, $f'(u) \geq a > 0$, and sufficiently small initial datum u_0 , the solution satisfies a gradient estimate of the type $\|\nabla u\|_p \leq Ce^{-\delta t} \|\nabla u_0\|_p$. This result was also established, under certain geometric conditions on $\partial\Omega$, for an initial boundary problem for the quasilinear equation of the form

$$(1.2) \quad u_t - \operatorname{div}(\sigma(|\nabla u|^2)\nabla u) + f(u) = 0.$$

Similar results concerning global existence and gradient estimates have been proved by Nakao and Ohara [8] and Nakao and Chen [9]. It is also worth mentioning that Pucci and Serrin [10] discussed the stability of the rest state for a quasilinear heat system of the form

$$A(t)|u_t|^{m-2}u_t = \Delta u - f(x, u),$$

for $m > 1$ and the source satisfying $(f(x, u), u) \geq 0$. They established a global result of solutions and showed that these solutions tend to the rest state as $t \rightarrow \infty$, however no rate of decay has been given.

In this work we consider

$$(1.3) \quad \begin{aligned} u_t - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) &= bu|u|^{p-2}, & x \in \Omega, & \quad t > 0 \\ u(x, t) &= 0, & x \in \partial\Omega, & \quad t \geq 0 \\ u(x, 0) &= u_0(x), & x \in \Omega, & \end{aligned}$$

$p > \alpha \geq 2$ and show that for suitably chosen initial data, (1.3) possesses a global weak solution, which decays exponentially for $\alpha = 2$ and in a polynomial rate if $\alpha > 2$. We first state an existence result, which can be established by repeating the same procedure of [4]. See also [1] and [3] for more standard results concerning local existence.

Proposition *Suppose that $p \geq 2$, such that*

$$(1.4) \quad \begin{aligned} 2 &\leq p \leq 1 + \frac{\alpha}{2} \frac{n}{n-\alpha}, & n &\geq \alpha \\ p &\geq 2, & n &< \alpha \end{aligned}$$

and let $u_0 \in W_0^{1,\alpha}(\Omega)$ be given. Then problem (1.3) has a unique solution

$$(1.5) \quad \begin{aligned} u &\in C\left([0, T]; W_0^{1,\alpha}(\Omega)\right) \\ u_t &\in L^2(\Omega \times (0, T)). \end{aligned}$$

for some T small.

2. MAIN RESULT

In order to state and prove our main result we remind that by the embedding theorem there exists a constant C_* depending on Ω, p and α only such that

$$(2.1) \quad \|u\|_p \leq C_* \|\nabla u\|_\alpha.$$

We also introduce the following

$$(2.2) \quad \begin{aligned} I(t) &= I(u(t)) = \|\nabla u(t)\|_\alpha^\alpha - b\|u(t)\|_p^p \\ E(t) &= E(u(t)) = \frac{1}{\alpha}\|\nabla u(t)\|_\alpha^\alpha - \frac{b}{p}\|u(t)\|_p^p \\ W &= \{w \in W_0^{1,\alpha}(\Omega) : I(w) > 0\} \cup \{0\}. \end{aligned}$$

Remark 2.1. By multiplying equation (1.3) by u_t , integrating over Ω , and using integration by parts, we get

$$(2.3) \quad E'(t) = -\|u_t(t)\|_2^2 \leq 0, \quad \forall t \in [0, T].$$

Lemma 2.1. Suppose that (1.4) holds. If $u_0 \in W$ satisfies

$$(2.4) \quad \beta = bC_*^p \left(\frac{\alpha p}{p - \alpha} E(u_0) \right)^{(p-\alpha)/\alpha} < 1$$

then $u(t) \in W$, for each $t \in [0, T)$.

Proof. Since $u_0 \in W$ then $I(u_0) > 0$. This implies the existence of $T_m \leq T$ such that $I(u(t)) \geq 0$ for all $t \in [0, T_m)$. This implies

$$(2.5) \quad \begin{aligned} E(t) &= \frac{1}{\alpha}\|\nabla u(t)\|_\alpha^\alpha - \frac{b}{p}\|u(t)\|_p^p \\ &= \frac{p - \alpha}{\alpha p}\|\nabla u(t)\|_\alpha^\alpha + \frac{1}{p}I(u(t)) \\ &\geq \frac{p - \alpha}{\alpha p}\|\nabla u(t)\|_\alpha^\alpha, \quad \forall t \in [0, T_m); \end{aligned}$$

hence

$$(2.6) \quad \|\nabla u(t)\|_\alpha^\alpha \leq \frac{\alpha p}{p-\alpha} E(t) \leq \frac{\alpha p}{p-\alpha} E(u_0), \quad \forall t \in [0, T_m].$$

By exploiting (2.1) and (2.6), we easily arrive at

$$(2.7) \quad \begin{aligned} b\|u(t)\|_p^p &\leq bC_*^p \|\nabla u(t)\|_\alpha^p = bC_*^p \|\nabla u(t)\|_\alpha^{p-\alpha} \|\nabla u(t)\|_\alpha^\alpha \\ &\leq bC_*^p \left(\frac{\alpha p}{p-\alpha} E(u_0) \right)^{(p-\alpha)/\alpha} \|\nabla u(t)\|_\alpha^\alpha = \beta \|\nabla u(t)\|_\alpha^\alpha \\ &< \|\nabla u(t)\|_\alpha^\alpha, \quad \forall t \in [0, T_m]; \end{aligned}$$

hence $\|\nabla u(t)\|_\alpha^\alpha - b\|u(t)\|_p^p > 0$, $\forall t \in [0, T_m]$. This shows that $u(t) \in W$, $\forall t \in [0, T_m]$. By repeating the procedure, T_m is extended to T .

Theorem 2.2. *Suppose that (1.4) holds. If $u_0 \in W$ satisfying (2.4) Then the solution is global*

Proof. It suffices to show that $\|\nabla u(t)\|_\alpha^\alpha$ is bounded independently of t . To achieve this we use (2.2) and (2.3)

$$(2.8) \quad \begin{aligned} E(u_0) &\geq E(t) = \frac{1}{\alpha} \|\nabla u(t)\|_\alpha^\alpha - \frac{b}{p} \|u(t)\|_p^p \\ &= \frac{p-\alpha}{\alpha p} \|\nabla u(t)\|_\alpha^\alpha + \frac{1}{p} I(u(t)) \geq \frac{p-\alpha}{\alpha p} \|\nabla u(t)\|_\alpha^\alpha \end{aligned}$$

since $I(u(t)) \geq 0$. Therefore

$$(2.9) \quad \|\nabla u(t)\|_\alpha^\alpha \leq \frac{\alpha p}{p-\alpha} E(u_0)$$

Theorem 2.3. *Suppose that (1.4) holds. Then there exist positive constants K and k such that, for all $t \geq 0$, the global solution of (1.3) satisfies*

$$(2.10) \quad \begin{aligned} E(t) &\leq K e^{-kt}, & \alpha &= 2 \\ E(t) &\leq (kt + K)^{-2/(\alpha-2)}, & \alpha &> 2. \end{aligned}$$

Proof. We define

$$(2.11) \quad H(t) := E(t) + \frac{1}{2} \int_\Omega u^2(t) dx,$$

hence we have $E(t) \leq H(t)$ and

$$\begin{aligned}
 (2.12) \quad H(t) &\leq E(t) + \frac{1}{2}C_*^2 \|\nabla u(t)\|_\alpha^2 \\
 &\leq E(t) + \frac{1}{2}C_*^2 \left(\frac{\alpha p}{p - \alpha} E(t) \right)^{2/\alpha} \\
 &\leq \left(E^{1-(2/\alpha)}(u_0) + \frac{1}{2}C_*^2 \left(\frac{\alpha p}{p - \alpha} \right)^{2/\alpha} \right) E^{2/\alpha}(t) \\
 &= cE^{2/\alpha}(t).
 \end{aligned}$$

We differentiate (2.11) and use equation (1.3) and (2.3) to obtain

$$(2.13) \quad H'(t) = - \int_\Omega |u_t(t)|^2 dx - \int_\Omega |\nabla u(t)|^\alpha dx + b \int_\Omega |u(t)|^p dx.$$

We then use (2.2) and (2.7) to get

$$\begin{aligned}
 b \int_\Omega |u(t)|^p dx &= \lambda b \int_\Omega |u(t)|^p dx + (1 - \lambda)b \int_\Omega |u(t)|^p dx \\
 &= \lambda \left(\frac{p}{\alpha} \int_\Omega |\nabla u(t)|^\alpha dx - pE(t) \right) \\
 (2.14) \quad &+ (1 - \lambda)\beta \int_\Omega |\nabla u(t)|^\alpha dx, \quad 0 < \lambda < 1.
 \end{aligned}$$

Therefore a combination of (2.13) and (2.14) gives

$$\begin{aligned}
 (2.15) \quad H'(t) &\leq - \int_\Omega u_t^2(t) dx - \lambda p E(t) \\
 &\quad + \varepsilon \left[\lambda \left(\frac{p}{\alpha} - 1 \right) - \eta(1 - \lambda) \right] \int_\Omega |\nabla u(t)|^\alpha dx,
 \end{aligned}$$

where $\eta = 1 - \beta$. By using (2.8) and choosing λ close to 1 so that

$$\lambda \left(\frac{p}{\alpha} - 1 \right) - \eta(1 - \lambda) > 0,$$

we arrive at

$$\begin{aligned}
 (2.16) \quad H'(t) &\leq - \int_\Omega u_t^2(t) dx - \lambda p E(t) \\
 &\quad + \left[\lambda \left(\frac{p}{\alpha} - 1 \right) - \eta(1 - \lambda) \right] \frac{\alpha p}{p - \alpha} E(t) \\
 &\leq - \int_\Omega u_t^2(t) dx - \eta(1 - \lambda) \frac{\alpha p}{p - \alpha} E(t).
 \end{aligned}$$

We then recall (2.12) to obtain, from (2.16),

$$(2.17) \quad H'(t) \leq -\eta(1-\lambda) \frac{\alpha p}{p-\alpha} c^{-\alpha/2} H^{\alpha/2}(t).$$

We distinguish two cases.

i) $\alpha = 2$, then a simple integration of (2.17) leads to

$$(2.18) \quad E(t) \leq H(t) \leq H(0)e^{-kt}, \quad \forall t \geq 0,$$

where $k = \frac{\eta}{c}(1-\lambda) \frac{\alpha p}{p-\alpha}$.

ii) $\alpha > 2$, again a simple integration of (2.17) yields

$$(2.19) \quad E(t) \leq H(t) \leq \left(kt + H^{(2-\alpha)/2}(0) \right)^{-2/(\alpha-2)},$$

where

$$k = \left(\frac{\alpha}{2} - 1 \right) \eta (1-\lambda) \frac{\alpha p}{p-\alpha} c^{-\alpha/2}.$$

This completes the proof.

Remark 2.2. By using (2.5), (2.8), (2.18), and (2.19), we easily obtain, for all $t \geq 0$,

$$(2.20) \quad \begin{aligned} \|\nabla u(t)\|_{\alpha} &\leq C e^{-kt/2}, & \alpha = 2 \\ \|\nabla u(t)\|_{\alpha} &\leq C (t+1)^{-2/(\alpha-2)\alpha}, & \alpha > 2 \end{aligned}$$

Acknowledgment. The author would like to thank KFUPM for its support.

REFERENCES

1. M. Chipot, *Element of Nonlinear Analysis*, Birkhauser Advanced Texts, Basler Lehrbucher, first Edition, 2000
2. H. Englern, B. Kawohl, and S. Luckhaus, *Gradient estimates for solutions of parabolic equations and systems*, J. Math. Anal. Appl., **147** (1990), 309 - 329
3. A. Friedman, *Partial differential equations of parabolic type*, Prentice Hall Englewood NJ 1964.

4. V. Georgiev and G. Todorova, *Existence of solutions of the wave equation with nonlinear damping and source terms*, J. Diff. Eqns., **109** (1994), 295 - 308.
5. V. K. Kalantarov and O. A. Ladyzhenskaya, *Formation of collapses in quasilinear equations of parabolic and hyperbolic types*. (Russian) Boundary value problems of mathematical physics and related questions in the theory of functions, 10. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **69** (1977), 77-102.
6. H. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$* , Archive Rat.Mech. Anal., **51** (1973), 371 - 386.
7. H. Levine, S. Park, and J. Serrin, *Global existence and nonexistence theorems for quasilinear evolution equations of formally parabolic type*, J. Diff. Eqns, **142** (1998), 212 - 229.
8. M. Nakao and Y. Ohara, *Gradient estimates for a quasilinear parabolic equation of the mean curvature type*, J. Math. Soc. Japan, **48** # 3 (1996), 455 - 466.
9. M. Nakao and C. Chen, *Global existence and gradient estimates for the quasilinear parabolic equations of m -Laplacian type with a nonlinear convection term*, J. Diff. Eqns, **162** (2000), 224 - 250.
10. P. Pucci and J. Serrin, *Asymptotic stability for nonlinear parabolic systems*, Energy methods in continuum mechanics, Oviedo, (1994), 66-74, Kluwer Acad. Publ., Dordrecht, 1996.

Mathematical Sciences Department, KFUPM, Dhahran 31261, Saudi Arabia.
e-mail: messaoud@kfupm.edu.sa

Date received Oct 29, 2003