Arab J. Math. Sc.
Volume 13, December 2007
pp.1-13

ON P - CLOSEDNESS IN A BITOPOLOGICAL SPACE

S. K. SEN AND M. N. MUKHERJEE

ABSTRACT. The object of this paper is to initiate the study of P-closedness in bitopological spaces. Several characterizations of such a concept have been formulated and proved by means of certain types of adherence and convergence of filters and nets.

1. INTRODUCTION AND PRELIMINARIES

It is found from the literature that different neighbouring forms of compactness in topological framework are being studied meticulously for a pretty long time by many mathematicians. The concept of p - closedness, one such covering property, was introduced by Abo - Khadra [1] in 1989 by use of preopen sets of Mashhour et al. [8]. Since then extensive investigations of such a concept have been pursued by a large number of topologists (for instance, see [2], [10], [11]). In view of all these investigations, p-closedness in topological spaces has assumed an important position in the hierarchy of covering properties.

It is well known that the introduction of bitopological space by Kelly [7] in 1963 as a generalization of topological structure has opened up a new direction for extensions of different important topological ideas to bitopological perspective, and numerous such attempts have so far been undertaken quite fruitfully. Our endeavour in this paper is to pursue the latter trend by extending the idea of P - closedness into bitopological setting.

AMS *subject classification* : **(2005) 54E55 , 54D99.**
Keywords and phrases: bitopological space, preopen set, $p(\theta)$-open set, $p(\theta)$-closure, $p(\theta)$-adherence and $p(\theta)$-convergence of filters and nets, P-closedness.

In the next section, we introduce the definition of $p(\theta)$-open sets, and $p(\theta)$-adherence and $p(\theta)$-convergence of filters and nets in the context of bitopological spaces. Certain descriptions of these ideas are also worked out, which facilitate our intended study in Section 3, concerning P-closedness of bitopological spaces.

Throughout the paper, by a space X we shall mean a bitopological space (X, Q_1, Q_2). Whenever i and j will occur simultaneously, we assume that $i, j \in \{1, 2\}$ and $i \neq j$. For a subset A of a space (X, Q_1, Q_2), Q_i-int$A$ and Q_i-cl$A$ will respectively stand for the interior and closure of A in (X, Q_i) (for $i = 1, 2$).We now recall some definitions and results which we shall need in the sequel.

Definition 1.1. Let (X, Q_1, Q_2) be a bitopological space and $A \subseteq X$. Then A is called ij-preopen [3] if $A \subseteq Q_i$-intQ_j-cl$A$; the complement of an ij-preopen set in X is called an ij-preclosed set or equivalently, $A(\subseteq X)$ is ij-preclosed iff $A \supseteq Q_i$-clQ_j-int$A$ ([4], [5]).

We shall denote the set of all ij-preopen subsets of X by $PO_{ij}(X)$ and that of all ij-preopen sets in X containing a point x of X by $PO_{ij}(x)$.

Definition 1.2. For a subset A of a space (X, Q_1, Q_2),
(i) the ij-preclosure of A [5], denoted by ij-pcl$A$, is defined by the intersection of all ij-preclosed sets containing A.
(ii) the ij-preinterior of A [12] is the union of all ij-preopen sets contained in A, to be denoted by ij-pint$A$.

Result 1.3. [4,5,12] In a bitopological space (X, Q_1, Q_2) the following hold:
(a) The union (intersection) of an arbitrary family of ij-preopen (ij-preclosed) sets is ij-preopen (resp. ij-preclosed).
(b) ij-pint A(ij-pcl A) is ij-preopen (resp. ij-preclosed), for any $A \subseteq X$.
(c) $A(\subseteq X)$ is ij-preopen (ij-preclosed) iff $A = ij$-pint$A$ (resp. $A = ij$-pcl$A$).
(d) For any $A(\subseteq X)$, $X \setminus ij$-pcl$A = ij$-pint$(X \setminus A)$.
(e) For any $A(\subseteq X)$ and any point $x \in X$, $x \in ij$-pcl$A$ ($x \in ij$-pint$A$) iff for all $U \in PO_{ij}(x)$, $U \cap A \neq \emptyset$ (resp. iff for some $U \in PO_{ij}(x)$, $U \subseteq A$).

Result 1.4. For any Q_j-open set A in a space (X, Q_1, Q_2), ij-pcl$A = Q_i$-cl$A$.

Proof. $x \in ij\text{-pcl}A \Rightarrow$ for all $U \in PO_{ij}(x)$, $U \cap A \neq \emptyset$. Since every Q_i-open set is ij-preopen, every Q_i-open set containing x always intersects A i.e., $x \in Q_i\text{-cl}A$. Again, $x \notin ij\text{-pcl}A \Rightarrow$ there exists $V \in PO_{ij}(x)$, such that $V \cap A = \emptyset \Rightarrow V \subseteq (X \setminus A) \Rightarrow Q_j\text{-cl}V \subseteq Q_j\text{-cl}(X \setminus A) = (X \setminus A)$ [since A is Q_j-open] $\Rightarrow Q_i\text{-int}Q_j\text{-cl}V \subseteq (X \setminus A)$. Now since V is an ij-preopen set containing x, $x \in V \subseteq Q_i\text{-int}Q_j\text{-cl}V$, so $Q_i\text{-int}Q_j\text{-cl}V$ is a Q_i-open set containing x and $Q_i\text{-int}Q_j\text{-cl}V \cap A = \emptyset$ and hence $x \notin Q_i\text{-cl}A$. □

2. ij-$p(\theta)$ OPEN SET, ij-$p(\theta)$ ADHERENCE AND CONVERGENCE OF FILTERS AND NETS.

Definition 2.1. For a subset A of a space (X, Q_1, Q_2), the ij-$p(\theta)$-closure of A, to be denoted by ij-$p(\theta)$-cl$A$, is defined by $ij\text{-}p(\theta)\text{-cl}A = \{x \in X : ji\text{-pcl}U \cap A \neq \emptyset, \forall U \in PO_{ij}(x)\}$.
A is called ij-$p(\theta)$-closed if $A = ij\text{-}p(\theta)\text{-cl}A$. A is called ij-$p(\theta)$-open if $X \setminus A$ is ij-$p(\theta)$-closed.

Theorem 2.2. *$A(\subseteq X)$ is ij-$p(\theta)$-open iff for each $x \in A$, there exists $U \in PO_{ij}(x)$ such that ji-pcl$U \subseteq A$.*

Proof. $x \in A \Rightarrow x \notin X \setminus A = ij\text{-}p(\theta)\text{-cl}(X \setminus A)$ [since A is ij-$p(\theta)$-open] $\Rightarrow$ there exists $U \in PO_{ij}(x)$ such that $ji\text{-pcl}U \cap (X \setminus A) = \emptyset$, i.e., $ji\text{-pcl}U \subseteq A$. Conversely, let $x \in X \setminus A$. Now there exists some $U \in PO_{ij}(x)$ such that $ji\text{-pcl}U \subseteq A$. So, $ji\text{-pcl}U \cap (X \setminus A) = \emptyset \Rightarrow x \notin ij\text{-}p(\theta)\text{-cl}(X \setminus A)$. Thus $X \setminus A = ij\text{-}p(\theta)\text{-cl}(X \setminus A)$,so that $X \setminus A$ is ij-$p(\theta)$-closed and hence A is ij-$p(\theta)$-open. □

Theorem 2.3. *An ij-$p(\theta)$-open set is ij-preopen and an ij-$p(\theta)$-closed set is ij-preclosed.*

Proof. Let A be an ij-$p(\theta)$-closed set in X. $A \subseteq ij\text{-pcl}A$ is obvious. Now, $x \notin A \Rightarrow x \notin ij\text{-}p(\theta)\text{-cl}A$ [since A is ij-$p(\theta)$-closed] $\Rightarrow$ there exists $U \in PO_{ij}(x)$ such that $ji\text{-pcl}U \cap A = \emptyset \Rightarrow U \cap A = \emptyset \Rightarrow x \notin ij\text{-pcl}A$. Again, let A be ij-$p(\theta)$-open, then $X \setminus A$ is ij-$p(\theta)$-closed, so $X \setminus A$ is also an ij-preclosed set and hence A is ij-preopen. □

In the following example, we show that the converse of the above theorem is false.

Example 2.4. Let $X = \{a, b, c\}$, $Q_1 = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $Q_2 = \{X, \emptyset, \{b\}, \{a, c\}\}$. Then $PO_{12}(X) = Q_1$. Now $\{b, c\}$ is ij-preopen but is not ij-$p(\theta)$-open.

Theorem 2.5. *For any ji-preopen set A in a space X, ij-$pclA = ij$-$p(\theta)$-clA.*

Proof. Clearly ij-pcl$A \subseteq ij$-$p(\theta)$-cl$A$. Now $x \notin ij$-pcl$A$ $\Rightarrow$ there exists $V \in PO_{ij}(x)$ such that $V \cap A = \emptyset \Rightarrow V \subseteq (X \setminus A) \Rightarrow ji$-pcl$V \subseteq ji$-pcl$(X \setminus A) = X \setminus A$ [since A is ji-preopen] $\Rightarrow ji$-pcl$V \cap A = \emptyset \Rightarrow x \notin ij$-$p(\theta)$-cl$A$. □

We recall that a point x of a space (X, Q_1, Q_2) is said to be in the ij-θ-closure of a subset A of X, in notation $x \in ij$-θ-cl$A$, if Q_j-cl$U \cap A \neq \emptyset$, for each Q_i-open neighbourhood (henceforth nbd, for short) U of x [6,9]. It is also known [6] that

Theorem 2.6. *For any Q_j-open set A in a space (X, Q_1, Q_2), Q_i-$clA = ij$-θ-clA.*

It now follows from Result 1.4 and Theorem 2.5 that

Corollary 2.7. *For any Q_j-open set A in (X, Q_1, Q_2),*
ij-$p(\theta)$-$clA = ij$-$pclA = Q_i$-$clA = ij$-θ-clA.

The following theorem gives an expression for the ij-$p(\theta)$-closure of any set.

Theorem 2.8. *For any subset A of a space (X, Q_1, Q_2),*

$$ij - p(\theta)\text{-}clA = \cap\{ij\text{-}pclU : A \subseteq U \in PO_{ji}(X)\}.$$

Proof. Let $U \in PO_{ji}(X)$ and $A \subseteq U$, then ij-$p(\theta)$-cl$A \subseteq ij$-$p(\theta)$-cl$U$ $= ij$-pcl$U \Rightarrow$ L.H.S $\subseteq$ R.H.S. Now, $x \notin ij$-$p(\theta)$-A $\Rightarrow$ there exists $V \in PO_{ij}(x)$ such that ji-pcl$V \cap A = \emptyset \Rightarrow A \subseteq X \setminus ji$-pcl$V = U$ (say) $\in PO_{ji}(X) \Rightarrow V \cap U = \emptyset$, and hence $x \notin ij$-pcl$U \Rightarrow x \notin$ R.H.S. □

Definition 2.9. Let (X, Q_1, Q_2) be a bitopological space, $A \subseteq X$ and $x \in X$.
(a) A filterbase $\Im$ in A is said to
(i) ij-$p(\theta)$-adhere at x (written as $x \in ij$-$p(\theta)$-ad$\Im$) if for each $U \in PO_{ij}(x)$

and each $F \in \Im$, $F \cap ji$-pcl$U \neq \emptyset$,
(ii) ij-$p(\theta)$-converge to x (written as $\Im \xrightarrow{ij-p(\theta)} x$) if for each $U \in PO_{ij}(x)$ there exists $F \in \Im$, such that $F \subseteq ji$-pcl$U$.
(b) A net $\{x_\alpha : \alpha \in (D, \geq)\}$ (where $(D, \geq)$ is a directed set) in A is said to
(i) ij-$p(\theta)$-adhere at x (written as $x \in ij$-$p(\theta)$-ad$\{x_\alpha\}$) if for each $U \in PO_{ij}(x)$ and each $\alpha \in D$, there exists a $\beta \in D$ with $\beta \geq \alpha$ such that $x_\beta \in ji$-pcl$U$,
(ii) ij-$p(\theta)$-converge to x (written as $x_\alpha \xrightarrow{ij-p(\theta)} x$) if for each $U \in PO_{ij}(x)$ there exists a $\beta \in D$ such that $x_\alpha \in ji$-pcl$U$ for all $\alpha \in D$ with $\alpha \geq \beta$.

Theorem 2.10. *Let (X, Q_1, Q_2) be a bitopological space and $x \in X$,*
(a) A filterbase $\Im$ on X, ij-$p(\theta)$-converges to $x_0(\in X)$ iff the net P based on $\Im$ is ij-$p(\theta)$-convergent to x_0.
(b) A net $\{x_\alpha\}$ in X ij-$p(\theta)$-converges to $x_0(\in X)$ iff the filterbase $\Im$ generated by the net $\{x_\alpha\}$ is ij-$p(\theta)$-convergent to x_0.

Proof. (a) Let a filterbase $\Im$ be ij-$p(\theta)$-convergent to x_0, and let $P : D \to X$ be the net based on $\Im$, where $D = \{(x, F) : x \in F \in \Im\}$ and $P[(x, F)] = x$. If $U \in PO_{ij}(X)$ and $x_0 \in U$, then for some $F \in \Im$, $F \subseteq ji$-pcl$U$. Choose $p \in F$ so that $(p, F) \in D$. If $(x_1, F_1) \geq (p, F)$, then $P[(x_1, F_1)] = x_1 \in F_1$ and as $F_1 \subseteq F$, $x_1 \in ji$-pcl$U$. i.e., P ,ij-$p(\theta)$-converges to x_0. Conversely, let P be ij-$p(\theta)$-convergent to x_0, and $U \in PO_{ij}(x_0)$. Then there is $(x_1, F_1) \in D$ such that $(y, F) \geq (x_1, F_1)$ implies $P[(y, F)] = y \in ji$-pcl$U$. Now for each $z \in F_1$ we have $(z, F_1) \geq (x_1, F_1)$. So, $F_1 \subseteq ji$-pcl$U$. Thus $\Im$ is ij-$p(\theta)$-convergent to x_0.
(b) Its proof is quite similar to above and hence is omitted. □

Theorem 2.11. *Let (X, Q_1, Q_2) be a bitopological space and $x \in X$,*
(a) A point x of X is an ij-$p(\theta)$-adherent point of a filterbase $\Im$ iff the net P based on $\Im$ has x as an ij-$p(\theta)$-adherent point.
(b) A point x of X is an ij-$p(\theta)$-adherent point of a net $\{x_\alpha\}$ iff x is an ij-$p(\theta)$-adherent point of the filterbase generated by $\{x_\alpha\}$.

Proof. (a) Let x be an ij-$p(\theta)$-adherent point of a filterbase $\Im$ and let $P : D \to X$ be the net based on $\Im$, where $D = \{(x, F) : x \in F \in \Im\}$ and $P[(x, F)] = x$. Let $x \in U \in PO_{ij}(x)$ and $(a, F) \in D$. If $b \in ji$-pcl$U \cap F$, then $(b, F) \geq (a, F)$ and $P[(b, F)] = b \in ji$-pcl$U$. Hence x is an ij-$p(\theta)$-adherent point of the net

P.

Conversely, let the net have x as an ij-$p(\theta)$-adherent point and $x \in U \in PO_{ij}(x)$. Let $F \in \Im$. If $a \in F$, then $(a, F) \in D$ and for some $(b, K) \in D$, $(b, K) \geq (a, F)$, $P[(b, K)] = b \in ji\text{-pcl}U$. As $K \subseteq F$, we have $F \cap ji\text{-pcl}U \neq \emptyset$. i.e., x is an ij-$p(\theta)$-adherent point of the filterbase $\Im$.

(b) We omit the straightforward proof which is similar to that of (a) above. □

Theorem 2.12. *A point x of a space X is an ij-$p(\theta)$-adherent point of a net $\{x_\alpha\}$ in X iff there exists a subnet $\{x_{\alpha_k}\}$ of $\{x_\alpha\}$ which ij-$p(\theta)$-converges to x.*

Proof. Let x be an ij-$p(\theta)$-adherent point of a net $\{x_\alpha\}$ with the directed set $(D, \geq)$ as the domain. Let $M = \{(\alpha, ji\text{-pcl}U) : x \in U \in PO_{ij}(x)$ and $x_\alpha \in ji\text{-pcl}U\}$. Define $(\alpha_1, ji\text{-pcl}U_1) \geq (\alpha_2, ji\text{-pcl}U_2)$ if and only if $\alpha_1 \geq \alpha_2$ in D and $ji\text{-pcl}U_1 \leq ji\text{-pcl}U_2$. Let $\psi : M \to D$ be defined by $\psi[(\alpha, ji\text{-pcl}U)] = \alpha$, so the map $(\alpha, ji\text{-pcl}U) \to x_\alpha$ with M as the domain defines a subnet of $\{x_\alpha\}$. Now if $x \in U \in PO_{ij}(x)$, then for some α, $x_\alpha \in ji\text{-pcl}U$ and if $(\beta, ji\text{-pcl}V) \geq (\alpha, ji\text{-pcl}U)$, then $x_\beta \in ji\text{-pcl}V \subseteq ji\text{-pcl}U$ and so the subnet ij-$p(\theta)$-converges to x.

Conversely, let $\{x_\alpha\}$ be a net with the directed set $(D, \geq)$ as the domain, and let $\{x_{\alpha_k}\}$ be a subnet of $\{x_\alpha\}$ with domain B, which ij-$p(\theta)$-converges to x. Let $x \in U \in PO_{ij}(x)$ and $\alpha_0 \in D$, take $\lambda_0 \in B$ such that $\alpha_{\lambda_0} \geq \alpha_0$, also let $\lambda_1 \in B$ such that if $\lambda \geq \lambda_1$, $x_\alpha \in ji\text{-pcl}U$. Let λ_2 be such that $\lambda_2 \geq \lambda_0$ and $\lambda_2 \geq \lambda_1$, then $\alpha_{\lambda_2} \geq \alpha_{\lambda_0} \geq \alpha_0$ and $x_{\alpha_{\lambda_2}} \in ji\text{-pcl}U$. Hence x is an ij-$p(\theta)$-adherent point of the net $\{x_\alpha\}$. □

Theorem 2.13. *A point x of X is an ij-$p(\theta)$-adherent point of a filterbase $\Im$ on X, iff there exists a filterbase $\Im^*$ finer than $\Im$, such that $\Im^* \xrightarrow{ij-p(\theta)} x$.*

Proof. Let $x \in ij\text{-}p(\theta)\text{-ad}\Im$. Let $\mathcal{U} = \{ji\text{-pcl}U : x \in U \in PO_{ij}(x)\}$. Then $\Im \cup \mathcal{U}$ forms a subbase for some filterbase $\Im^* \supseteq \Im$ such that $\Im^* \xrightarrow{ij-p(\theta)} x$. The converse is obvious. □

Corollary 2.14. *An ultrafilter $\mathcal{U}$ ij-$p(\theta)$-converges to $x \in X$ iff $x \in ij$-$p(\theta)$-ad$\mathcal{U}$.*

3. PAIRWISE P-CLOSEDNESS

Definition 3.1. A non-void subset A of a space (X, Q_1, Q_2) is said to be ij-P-closed relative to X if for every cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by ij-preopen sets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteq \{ji\text{-pcl}U_\alpha : \alpha \in \Lambda_0\}$. If in addition $A = X$, then X is called an ij-P-closed space. If A(or X) is ij-P-closed for $i, j = 1, 2$. $i \neq j$ then A(or X) is called pairwise P-closed.

Theorem 3.2. *Let (X, Q_1, Q_2) be a bitopological space, and $A \subseteq X$ be non-void, then the following are equivalent:*
(a) A is ij-P-closed relative to X.
(b) Every maximal filterbase on X which meets A, ij-$p(\theta)$-converges to some point of A.
(c) Every filterbase on X which meets A, ij-$p(\theta)$-adheres at some point of A.
(d) For every family $\{U_\alpha : \alpha \in \Lambda\}$ of non-void ij-preclosed sets of X with $(\cap U_\alpha) \cap A = \emptyset$, there is a finite subset Λ_0 of Λ such that $\cap\{ji\text{-pint}U_\alpha : \alpha \in \Lambda_0\} \cap A = \emptyset$.
(e) Every maximal filterbase on A is ij-$p(\theta)$-convergent to some point of A.
(f) Every filterbase on A, ij-$p(\theta)$-adheres at some point of A.
(g) For every family $\{B_\alpha : \alpha \in \Lambda\}$ of non-empty sets in X with $[\bigcap_{\alpha \in \Lambda} ij\text{-}p(\theta)\text{-cl}B] \cap A = \emptyset$, there exists a finite subset Λ_0 of Λ such that $[\bigcap_{\alpha \in \Lambda_0} B_\alpha] \cap A = \emptyset$.
(h) Every net in A, ij-$p(\theta)$-adheres at some point in A.
(i) Every ultranet in A, ij-$p(\theta)$-converges to some point of A.
(j) Every net in A has an ij-$p(\theta)$-convergent subnet .

Proof. **(a)⇒(b):** Let $\Im$ be a maximal filterbase on X, which meets A. Suppose that $\Im$ does not ij-$p(\theta)$-converge to any point of A. Since $\Im$ is a maximal filterbase (by Corollary 2.14), $\Im$ does not ij-$p(\theta)$-adhere at any point of A. For each $x \in A$, there exist $F_x \in \Im$ and $V_x \in PO_{ij}(x)$ such that $ji\text{-pcl}V_x \cap F_x = \emptyset$. The family $\{V_x : x \in A\}$ is a cover of A, by ij-preopen sets of X. By (a), there exists a finite number of points $x_1, x_2, x_3, \ldots, x_n$ of A such that $A \subseteq \bigcup\{ji\text{-pcl}V_{x_i} : i = 1, 2, 3, \ldots, n\}$. Since $\Im$ is a filterbase on X, there exists $F_0 \in \Im$ such that $F_0 \subseteq \cap\{F_{x_i} : i = 1, 2, 3, \ldots, n\}$. Therefore we obtain $F_0 \cap A = \emptyset$. This is a contradiction.

(b)⇒(c): Let $\Im$ be any filterbase on X, which meets A. Then there exists a maximal filterbase $\Im^*$ such that $\Im \subseteq \Im^*$, By (b), $\Im^*$ ij-$p(\theta)$-converges to some point $x \in A$. For every $F \in \Im$ and every $V \in PO_{ij}(x)$ there exists $F^* \in \Im^*$ such that $F^* \subseteq ji$-pcl$V$, hence $\emptyset \neq F^* \cap F \subseteq ji$-pcl$V \cap F$. This shows that $\Im$ ij-$p(\theta)$−adheres at x.

(c)⇒(d): Let $\{V_\alpha : \alpha \in \Lambda\}$ be any family of ij-preclosed subsets of X such that $\cap\{V_\alpha : \alpha \in \Lambda\} \cap A = \emptyset$. Assume that $\cap\{ji\text{-pint}V_\alpha : \alpha \in \Lambda_0\} \cap A \neq \emptyset$, for every finite subset Λ_0 of Λ. Then the family $\Im = \{\cap\{ji\text{-pint}V_\alpha\} : \Lambda_0 \text{ is a finite subset of } \Lambda\}$ is a filterbase on X which meets A. By (c), $\Im$ ij-$p(\theta)$-adheres at some point $x \in A$. Now, $\{A \setminus V_\alpha : \alpha \in \Lambda\}$ is a cover of A and so $x \in A \setminus V_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Therefore, we obtain $X \setminus V_{\alpha_0} \in PO_{ij}(x)$ and ji-pint$V_{\alpha_0} \in \Im$ and ji-pcl$(X \setminus V_{\alpha_0}) \cap ji$-pint$V_{\alpha_0} = \emptyset$. This is a contradiction (by Result 1.3(d)).

(d)⇒(a): Let $\{V_\alpha : \alpha \in \Lambda\}$ be any cover of A by ij-preopen sets of X. Then $\{X \setminus V_\alpha : \alpha \in \Lambda\}$ is a family of ij-preclosed subsets of X such that $\cap\{X \setminus V_\alpha : \alpha \in \Lambda\} \cap A = \emptyset$. By (d) there exists a finite subset Λ_0 of Λ such that $\bigcap_{\alpha \in \Lambda_0} \{ji\text{-pint}(X \setminus V_\alpha)\} \cap A = \emptyset$. Hence $A \subseteq \cup\{ji\text{-pcl}V_\alpha : \alpha \in \Lambda_0\}$. Thus A is ij-P-closed .

(c)⇒(f): It is obvious.

(b)⇔(e): Let $\Im$ be a maximal filterbase on X which meets A. Then $F^* = \{F \cap A : F \in \Im\}$ is a maximal filterbase on A and hence the rest is obvious.

(f)⇒(g): Let $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda\}$ be a family of non-void sets in X such that for every finite subset Λ_0 of Λ, $(\bigcap_{\alpha \in \Lambda_0} B_\alpha) \cap A \neq \emptyset$. Then $\Im = \{(\bigcap_{\alpha \in \Lambda_0} B_\alpha) \cap A : \Lambda_0 \text{ is a finite subset of } \Lambda\}$ is a filterbase on A. By (f) let $a \in A \cap (ij\text{-}p(\theta)\text{-ad}\Im)$. Then for each $\alpha \in \Lambda$ and each ij-preopen set U containing a, $A \cap B_\alpha \cap (ji\text{-pcl}U) \neq \emptyset$. Hence $a \in ij$-$p(\theta)$-clB_α for each $\alpha \in \Lambda$ and consequently, $(\bigcap_{\alpha \in \Lambda} ij\text{-}p(\theta)\text{-cl}B_\alpha) \cap A \neq \emptyset$.

(f)⇔(h): Follows from 2.11.

(h)⇔(j): Follows from 2.12.

(h)⇔(i): Follows from 2.11, since every net in A has a subnet which is an ultranet.

(g)⇒(a): Let $\{U_\alpha : \alpha \in \Lambda\}$ be any cover of A by ij-preopen sets of X. Then $A \cap (\bigcap_{\alpha \in \Lambda} (X \setminus U_\alpha)) = \emptyset$. If for some $\alpha \in \Lambda$, $X \setminus ji\text{-pcl}U_\alpha = \emptyset$, then the proof is complete. So we take $X \setminus ji\text{-pcl}U_\alpha$ ($= B_\alpha$, say) $\neq \emptyset$ for all $\alpha \in \Lambda$. Then $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda\}$ is a family of non-empty sets such that $[\bigcap_{\alpha\in\Lambda} ij\text{-}p(\theta)\text{-cl}B_\alpha] \cap A \subseteq A \cap [\cap(X \setminus U_\alpha)] = \emptyset \cdots (i)$. In fact, let $x \in ij\text{-}p(\theta)\text{-cl}B_\alpha = ij\text{-}p(\theta)\text{-cl}(X\ ji\text{-pcl}U_\alpha)$. Then for every ij-preopen set V_x containing x, $(X \setminus ji\text{-pcl}U_\alpha) \cap ji\text{-pcl}V_x \neq \emptyset$. Since $U_\alpha \in PO_{ij}(X)$, if $x \in U_\alpha$, then $(X \setminus ji\text{-pcl}U_\alpha) \cap ji\text{-pcl}U_\alpha \neq \emptyset$, which is not possible. Thus $x \notin U_\alpha$ i.e., $x \in X \setminus U_\alpha$ i.e., $ij\text{-}p(\theta)\text{-cl}B_\alpha \subseteq X \setminus U_\alpha$ and (i) follows. Now by (g), there is a finite subset Λ_0 of Λ such that $(\bigcap_{\alpha\in\Lambda_0} B_\alpha) \cap A = \emptyset$ i.e., $A \subseteq X \setminus [\bigcap_{\alpha\in\Lambda_0} (X\ ji\text{-pcl}U_\alpha)] = \bigcup_{\alpha\in\Lambda_0} ji\text{-pcl}U_\alpha$. Thus A is ij-P-closed. □

Putting $A = X$ in the above theorem, we obtain the following characterizations of an ij-P-closed space.

Theorem 3.3. *For a bitopological space (X, Q_1, Q_2), the following are equivalent:*

(a) X is ij-P-closed,

(b) Every maximal filterbase on X, ij-$p(\theta)$-converges.

(c) Every filterbase on X, ij-$p(\theta)$-adheres at some point in X.

(d) For every family $\{U_\alpha : \alpha \in \Lambda\}$ of non-void ij-preclosed sets of X with $(\cap U_\alpha) = \emptyset$, there is a finite subset Λ_0 of Λ such that $\cap\{ji\text{-pint}U_\alpha : \alpha \in \Lambda_0\} = \emptyset$.

(e) For every family $\{B_\alpha : \alpha \in \Lambda\}$ of non empty sets in X with $[\bigcap_{\alpha\in\Lambda} ij\text{-}p(\theta)\text{-cl}B_\alpha] = \emptyset$, there exists a finite subset Λ_0 of Λ such that $[\bigcap_{\alpha\in\Lambda_0} B_\alpha] = \emptyset$.

(f) Every net in X ij-$p(\theta)$-adheres at some point in X.

(g) Every ultranet in X ij-$p(\theta)$-converges.

(h) Every net in X has an ij-$p(\theta)$-convergent subnet.

Theorem 3.4. *In an ij-P-closed space (X, Q_1, Q_2), every family of ij-$p(\theta)$-closed sets with finite intersection property has non-void intersection.*

Proof. Let $\mathcal{U}$ be a family of ij-$p(\theta)$-closed sets. If $\cap\,\mathcal{U} = \emptyset$, then for $x \in X$, there exists $U_x \in \mathcal{U}$, such that $x \notin U_x$. Thus there exists a $V_x \in PO_{ij}(x)$ such that $ji\text{-pcl}V_x \cap U_x = \emptyset$. Now $\mathcal{V} = \{V_x : x \in X\}$ is a cover of X by

ij-preopen sets, so that there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X with $X = \bigcup_{i=1}^{n} ji\text{-pcl}V_x$. Again $(\bigcap_{i=1}^{n} U_x) \cap (\bigcup_{i=1}^{n} ji\text{-pcl}V_x) = \emptyset$. Thus $\bigcap_{i=1}^{n} U_x = \emptyset$, which contradicts the finite intersection property of $\mathcal{U}$. □

Theorem 3.5. *A space X is ij-P-closed iff every filterbase on X with at most one ij-$p(\theta)$-adherent point ij-$p(\theta)$-converges.*

Proof. Let X be ij-P-closed and $\Im$ be a filterbase on X with at most one ij-$p(\theta)$-adherent point. By Theorem 3.3, $\Im$ has then a unique ij-$p(\theta)$-adherent point x_0 (say). Let $\Im$ do not ij-$p(\theta)$-converge to x_0. Then for some $U \in PO_{ij}(x_0)$ and for each $F \in \Im$, $F \cap (X \setminus ji\text{-pcl}U) \neq \emptyset$. So $\mathcal{G} = \{F \cap (X \setminus ji\text{-pcl}U) : F \in \Im\}$ is a filterbase on X and hence ij-$p(\theta)$-adheres at some point x in X. Since $U \in PO_{ij}(x_0)$, $(ji\text{-pcl}U) \cap G = \emptyset$ for all $G \in \mathcal{G}$, we have $x \neq x_0$. Now for each $V \in PO_{ij}(x)$ and each $F \in \Im$, $F \cap (ji - \text{cl}V) \supseteq [F \cap (ji\text{-cl}V)] \cap [X \setminus ji\text{-cl}U] \neq \emptyset$ i.e., $F \cap (ji\text{-cl}V) \neq \emptyset$. Thus x is an ij-$p(\theta)$-adherent point of $\Im$, where $x \neq x_0$, a contradiction.
The converse is clear, by theorem 3.3((c) $\Rightarrow$ (a)), since x is necessarily an ij-$p(\theta)$-adherent point of a filterbase $\Im$ if $\Im \xrightarrow{ij-p(\theta)} x$. □

Theorem 3.6. *If X is ij-P-closed, then every cover of X by ij-$p(\theta)$-open sets has a finite subcover.*

Proof. Suppose $\mathcal{U}$ is a cover of an ij-P-closed space X by ij-$p(\theta)$-open sets. For each $x \in X$, $x \in U_x$ for some $U_x \in \mathcal{U}$. Then by Theorem 2.2 there exists a $V_x \in PO_{ij}(x)$ such that $ji\text{-pcl}V_x \subseteq U_x$. Thus $\{V_x : x \in X\}$ is a cover of X by ij-preopen sets. Now since X is ij-P-closed, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $X = \bigcup_{i=1}^{n} ji\text{-pcl}V_{x_i}$. Then $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. □

Definition 3.7. A family $\mathcal{U}$ of ji-preclosed subsets of a space X will be called an ij-precover of X, if for each $x \in X$, there is some $U \in \mathcal{U}$ such that U is an ij-prenbd of x (i.e., $x \in V \subseteq U$, for some ij-preopen set V).

Theorem 3.8. *A space X is ij-P-closed iff every ij-precover of X has a finite subcover.*

Proof. Let $\mathcal{U}$ be an ij-precover of an ij-P-closed space X. Then for each $x \in X$, there exist a $U_x \in \mathcal{U}$ and an ij-preopen set V_x such that $x \in V_x \subseteq U_x$. It then follows that $\{V_x : x \in X\}$ is a cover of X by ij-preopen sets of X. By ij-P-closedness of X, $X = \bigcup\limits_{i=1}^{n} ji\text{-pcl}V_{x_i} \subseteq \bigcup\limits_{i=1}^{n} ji\text{-pcl}U_{x_i} = \bigcup\limits_{i=1}^{n} U_{x_i}$.
Conversely, if $\mathcal{U}$ is any ij-preopen cover of X, then $\{ji\text{-pcl}U : U \in \mathcal{U}\}$ is an ij-precover of X. Hence the result. □

Theorem 3.9. *In an ij-P-closed space X, an ij-$p(\theta)$-closed set is ij-P-closed relative to X.*

Proof. Let $\mathcal{U}$ be an ij-preopen cover of an ij-$p(\theta)$-closed set A. Then for each $x \notin A$, there exists $V_x \in PO_{ij}(x)$ such that $ji\text{-pcl}V_x \cap A = \emptyset$. Now $\mathcal{U} \cup \{V_x : x \notin A\}$ is a cover of X by ij-preopen sets. Since X is ij-P-closed, $A \subseteq \bigcup\limits_{U \in \mathcal{U}} ji\text{-pcl}U$, for some finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$. Hence A is ij-P-closed relative to X. □

Definition 3.10. [13] A bitopological space (X, Q_1, Q_2) is said to be pairwise Urysohn if for any two distinct points x, y of X, there exist a Q_i-open nbd U of x and a Q_j-open nbd V of y such that $Q_j\text{-cl}U \cap Q_i\text{-cl}V = \emptyset$.

Theorem 3.11. *Let X be a pairwise Urysohn space, and $A(\subseteq X)$ be ij-P-closed relative to X, then A is ji-$p(\theta)$-closed.*

Proof. Let X be a pairwise Urysohn space, and $A(\subseteq X)$ be ij-P-closed relative to X and let $x \notin A$. Then for each $a \in A$, there exist $W_a \in Q_i$ and $V_x^a \in Q_j$ such that $x \in V_x^a$, $a \in W_a$ and $Q_i\text{-cl}V_x^a \cap Q_j\text{-cl}W_a = \emptyset$, i.e., $ij\text{-pcl}V_x^a \cap ji\text{-pcl}W_a = \emptyset$, (by Result 1.4). As A is ij-P-closed relative to X, and $\{W_a : a \in A\}$ covers A [$W_a \in Q_i \Rightarrow W_a$ is also an ij-preopen set], we have $A \subseteq \bigcup\limits_{r=1}^{n} ji\text{-pcl}W_{a_r}$ for finitely many $a_1, a_2, \ldots, a_n \in A$. Now let $V = \bigcap\limits_{r=1}^{n} V_x^{a_r}$, then V is also a ji-preopen set containing x, and $ij\text{-pcl}V \cap A = \emptyset$. Hence A is ij-$p(\theta)$-closed. □

Theorem 3.12. *Let A, B be two subsets of X. If A is ij-$p(\theta)$-closed and B is ij-P-closed relative to X, then $A \cap B$ is ij-P-closed relative to X.*

Proof. Let $\{V_\alpha : \alpha \in A\}$ be any cover of $A \cap B$ by ij-preopen sets of X. Since A is ij-$p(\theta)$-closed, for each $x \in B \setminus A$, there exists $W_x \in PO_{ij}(x)$ such that ji-pcl$W_x \cap A = \emptyset \Rightarrow ji$-pcl$W_x \subseteq X \setminus A$. The family $\{W_x : x \in B \setminus A\} \cup \{V_\alpha : \alpha \in A\}$ is a cover of B by ij-preopen sets of X. Since B is ij-P-closed relative to X, there exist a finite number of points $x_1, x_2, \ldots, x_n$ in $B \backslash A$, and a finite subset A_0 of A, such that $B \subseteq \{\bigcup_{i=1}^{n} ji\text{-pcl}W_{x_i}\} \cup \{\bigcup_{\alpha \in A_0} ji\text{-pcl}V_\alpha\}$. Since ji-pcl$W_{x_i} \cap A = \emptyset$ for each i, we obtain $A \cap B \subseteq \bigcup_{\alpha \in A_0} ji\text{-pcl}V_\alpha$. Hence $A \cap B$ is ij-P-closed relative to X. □

References

[1] Abd.EI-Aziz Ahmed Abo-Khadra, On generalized forms of compactness, *Master's thesis, Faculty of Science, Tanta University, Egypt.*(1989).

[2] J. Dontchev, M. Ganster and T. Noiri, On p-closed spaces, *Internat.Jour. Math. Math. Sci.* 24(3) (2000), 203-212.

[3] M.Jelić, A decomposition of pairwise continuity, *J. Inst. Math. Comput. Sci. Math. Ser.* 3(1990), 25 - 29.

[4] A. Kar and P.Bhattacharyya, Bitopological preopen sets, precontinuity and preopen mappings, *Indian J. Math.* 34(1992), 295 - 309.

[5] F.H.Khedr, S.M.Al-Areefi and T.Noiri, Precontinuity and semi- precontinuity in bitopological spaces, *Indian J. Pure Appl. Math.* 23(1992), 625 - 633.

[6] C. G. Kariofillis, On pairwise almost compactness, *Ann. Soc. Sci. Bruxelles* 100(1986), 129-137.

[7] J. C. Kelly, Bitopological spaces. *Proc.London Math. Soc.*(3) 13(1963), 71-89.

[8] A. S. Mashhour, M. E .Abd EI-Monsef and S.N.EI-Deeb, On precontinuous and weakly precontinuous mappings, *Proc. Math. Phys. Soc. Egypt* 53(1982), 47-53.

[9] M. N. Mukherjee,On pairwise almost compactness and pairwise H-closedness in a bitopological space, *Ann. Soc. Sci. Bruxelles* 96(1982), 98-106.

[10] M. N. Mukherjee, A. Debray and P. Sinha, p-closedness in topological spaces, *Revista De La Academia Canaria De Ciencias* XII(Nos. 1-2)(2000), 33-50.

[11] M. N. Mukherjee, B. Roy and P. Sinha, Concerning p-closed topological spaces, *Revista De La Academia Canaria De Ciencias* XIV(Nos. 1-2)(2002), 9-23.

[12] T.Noiri and V. Popa, On weakly precontinuous functions in bitopological spaces, *Soochow J. Math.* 33(2007) , 87 - 100.

[13] T. G. Raghavan and I.L.Reilly, On minimal bitopological spaces, *Kyungpook Math. J.* 17(No.1) (1977), 57-65.

[14] M. K. Singal and A Mathur, On nearly compact spaces, *Boll. Un. Mat. Ital.* 4(6)(1969), 702-710.

S. K. Sen.
Acharya Prafulla Chandra College,
New Barracpore, 24 parganas (N),
Kolkata-700131, West Bengal
INDIA
E-mail: apc1960@dataone.in

M. N. Mukherjee.
Department of Pure Mathematics,
University of Calcutta,
35, Ballygunge Circular Road,
Kolkata - 700019, West Bengal,
INDIA.
E-mail: mukherjeemn@yahoo.co.in

Date received October 30, 2007