Arab J. Math. Sc.
Volume 14, Number 1, June 2008
pp 1-14

A NOTE ON TRANSVERSAL HYPERSURFACES OF ALMOST HYPERBOLIC CONTACT MANIFOLDS

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ABSTRACT. Transversal hypersurfaces of trans hyperbolic contact manifolds are studied. It is proved that transversal hypersurfaces of almost hyperbolic contact manifold admits an almost product structure and each transversal hypersurfaces of almost hyperbolic contact metric manifold admits an almost product semi-Riemannian structure. The fundamental 2-form on the transversal hypersurfaces of cosymplectic hyperbolic manifold and $(\alpha, 0)$ trans hyberbolic Sasakian manifold with hyperbolic (f, g, u, v, λ)-structure are closed. It is also proved that transversal hypersurfaces of trans hyperbolic contact manifold admits a product structure. Some properties of transversal hypersurfaces are proved.

1. INTRODUCTION

Almost contact metric manifold with an almost contact metric structure is very well explained by Blair [1]. In [20], S. Tanno gave a classification for connected almost contact metric manifolds whose automorphism groups have the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes : (1) Homogenous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and

2000 AMS subject classification. 53C40
Key words. Almost hyperbolic contact metric manifolds, Trans hyperbolic contact manifolds, Almost product semi-Riemannian manifold structure, Transversal hypersurfaces of cosymplectic hyperbolic manifold, Transversal hypersurfaces of trans hyperbolic Sasakian manifolds.

(3) a warped product space $R \times_f C^n$ if $c < 0$. It is known that the manifolds of class (1) are characterized by some tensorial relations admitting a contact structure. Kenmotsu [10] characterized the differential geometric properties of the third case by tensor equation$(\overline{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$. The structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [10].

Oubina studied a new class of almost contact Riemannian manifold known as trans-Sasakian manifold [8] which generalizes both α-Sasakian [8] and β-Kenmotsu [8] structure.

M. D. Upadhayay studied almost contact hyperbolic (f, g, η, ξ) – structure [21]. Bhatt and Dubey studied on CR-submanifolds of trans hyperbolic contact manifold [3]. B. Y. Chen studied Geometry of submanifolds and its applications. Sci. Univ Tokyo. Tokyo, 1981. [5]. R. Prasad, M. M. Tripathi, J. S. Kim and J-H. Cho., studied some properties of submanifolds of almost contact manifold [15], [16], [17], [18], [19].

Let $\overline{M}$ be an $2n+1$ dimensional manifold with almost hyperbolic contact metric structure (ϕ, ξ, η, g) where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the semi Riemannian metric on $\overline{M}$. Then the following conditions [21] are satisfied

$$\phi^2 X = X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta o \phi = 0, \quad \eta(\xi) = -1, \tag{1.1}$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X).\eta(Y), \tag{1.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{1.3}$$

for vector fields X, Y on $\overline{M}$. An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called trans hyperbolic contact [3] if and only if

$$(\overline{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{1.4}$$

for all smooth vector fields X, Y on $\overline{M}$ and α, β non zero constant, where $\overline{\nabla}$ is the Levi-civita connection with respect to g. From (1.4) it follows that

$$\overline{\nabla}_X \xi = \alpha\phi X + \beta\left(X + \eta\left(X\right)\xi\right), \tag{1.5}$$

for all smooth vector fields X, Y on $\overline{M}$.

2. Transversal hypersurface

Let M be a hypersurface of an almost hyperbolic contact manifold $\overline{M}$ equipped with an almost hyperbolic contact structure $((\phi, \xi, \eta)$. We assume that the structure vector field ξ never belongs to tangent space of the hypersurface M, such that a hypersurface is called a transversal hypersurface of an almost contact manifold. In this case the structure vector field ξ can be taken as an affine normal to the hypersurface. Vector field X on M and ξ are linearly independent, therefore we may write

$$\phi X = F\left(X\right) + \omega\left(X\right)\xi \tag{2.1}$$

where F is a $(1, 1)$ tensor field and ω is a 1-form on M.

From (2.1)

$$\phi\xi = F\xi + \omega\left(\xi\right)\xi$$

or,

$$0 = F\xi + \omega\left(\xi\right)\xi$$

$$\phi^2 X = F\left(\phi X\right) + \omega\left(\phi X\right)\xi \tag{2.2}$$

$$X + \eta\left(X\right)\xi = F\left(FX + \omega\left(X\right)\xi + \omega\left(FX + \omega\left(X\right)\xi\right)\xi\right)$$

$$X + \eta\left(X\right)\xi = F^2 X + \left(\omega \circ F\right)\left(X\right)\xi \tag{2.3}$$

Taking account of equation (2.3), we get

$$F^2X = X \tag{2.4}$$

$$F^2 = I \tag{2.5}$$

$$\eta = \omega \circ F$$

Thus we have

Theorem 2.1. Each transversal hypersurface of an almost hyperbolic contact manifold admits an almost product structure and a 1-form ω.

From (2.4) and (2.5), it follows that

$$\eta = \omega \circ F$$

$$\eta(FX) = (\omega \circ F) FX$$

$$\eta(FX) = \omega\left(F^2X\right)$$

$$(\omega \circ F) X = \omega(X)$$

$$\omega = \eta \circ F \tag{2.6}$$

Now, we assume that $\overline{M}$ admits an almost hyperbolic contact metric structure (ϕ, ξ, η, g). We denote by g the induced metric on M also. Then for all $X, Y \in TM$, we obtain

$$g(FX, FY) = -g(X,Y) - \eta(X).\eta(Y) + \omega(X)\omega(Y) \tag{2.7}$$

We define a new metric G on the transversal hypersurface given by

$$G(X,Y) = g(\phi X, \phi Y) = -g(X,Y) - \eta(X).\eta(Y). \tag{2.8}$$

So,

$$\begin{aligned} G(FX, FY) &= -g(FX, FY) - \eta(FX).\eta(FY) \\ &= -g(X,Y) - \eta(X).\eta(Y) + \omega(X)\omega(Y) - (\eta \circ F)(X)(\eta \circ F)(Y) \\ &= -g(X,Y) - \eta(X).\eta(Y) + \omega(X)\omega(Y) - \omega(X)\omega(Y) \\ &= -g(X,Y) - \eta(X).\eta(Y) = G(X,Y) \end{aligned}$$

Then, we get

$$G(FX, FY) = G(X,Y), \tag{2.9}$$

where equation (2.4), (2.6), (2.7) and (2.8) are used.

Then G is semi Riemannian metric on M. that is (F,G) is an almost product semi-Riemannian structure on the transversal hypersurface M of $\overline{M}$. Thus, we are able to state the following.

Theorem 2.2. Each transversal hypersurface of an almost hyperbolic contact manifold admits an almost product semi-Riemannian structure. We now assume that M is orientable and choose a unit vector field N of $\overline{M}$, normal to M. Then Gauss and Weingarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)N, \quad (X,Y \in TM) \tag{2.10}$$

$$\overline{\nabla}_X N = -HX \tag{2.11}$$

where $\overline{\nabla}$ and ∇ are respectively the Levi-civita and induced Levi-civita connections in $\overline{M}$, M and h is the second fundamental form related to H by

$$h(X,Y) = g(HX,Y), \tag{2.12}$$

for any vector field X tangent to M, defining

$$\phi X = fX + u(X)N, \tag{2.13}$$

$$\phi N = -U, \tag{2.14}$$

$$\xi = V + \lambda N, \tag{2.15}$$

$$\begin{aligned} \eta(X) &= v(X), \\ \lambda &= \eta(N) = g(\xi, N), \end{aligned} \tag{2.16}$$

for $X \in TM$ we get an induced hyperbolic (f,g,u,v,λ)-structure on the transversal hypersurface such that

$$f^2 = I + u \otimes U + v \otimes V \tag{2.17}$$

$$fU = -\lambda V, \quad fV = \lambda U \tag{2.18}$$

$$u \circ f = \lambda v, \quad v \circ f = -\lambda U \tag{2.19}$$

$$u(U) = -1 - \lambda^2, \quad u(V) = v(U) = 0, \quad v(V) = -1 - \lambda^2 \tag{2.20}$$

$$g(fX, fY) = -g(X,Y) - u(X)u(Y) - v(X)v(Y) \tag{2.21}$$

$$g(X, fY) = -g(fX, Y), \quad g(X,U) = u(X), \quad g(X,V) = v(X), \tag{2.22}$$

for all $X, Y \in TM$, where

$$\lambda = \eta(N). \tag{2.23}$$

Thus, we see that every transversal hypersurface of an almost hyperbolic contact metric manifold also admits a hyperbolic (f, g, u, v, λ)-structure. Next we find relation between the induced almost product structure (F, G) and the induced hyperbolic (f, g, u, v, λ)-structure on the transversal hypersurface of an almost hyperbolic contact metric manifold. In fact, we have the following

Theorem 2.3. Let M be a transversal hypersurface of an almost hyperbolic contact metric manifold $\overline{M}$ equipped with almost hyperbolic contact metric structure (ϕ, ξ, η, g) and induced almost product structure (F, G).

Then we have

$$\lambda w = u, \tag{2.24}$$

$$F = f - \frac{1}{\lambda} u \otimes V, \tag{2.25}$$

$$FU = \frac{1}{\lambda} V, \tag{2.26}$$

$$uoF = uof = \lambda v, \tag{2.27}$$

$$FV = fV = \lambda U, \tag{2.28}$$

$$uoF = \frac{1}{\lambda} u. \tag{2.29}$$

Proof.

$$\phi X = FX + \omega(X)\xi,$$
$$\xi = V + \lambda N,$$

$$\phi X = FX + \omega(X) V + \lambda \omega(X) N, \tag{2.30}$$

$$\phi X = fX + u(X) N. \tag{2.31}$$

From equation (2.30) and (2.31) we have

$$\lambda \omega X = u(X), \qquad \omega(X) = \frac{1}{\lambda} u(X),$$

$$FX = fX - \omega(X)V,$$
$$FX = fX - \frac{1}{\lambda}u(X)V,$$
$$F = f - \frac{1}{\lambda}u \otimes v,$$

which is equation (2.25).

$$(uoF)(X) = (uof)(X) - \frac{1}{\lambda}u(X)u(V), \qquad u(V) = 0,$$
$$uoF = uof = \lambda v,$$

which is equation (2.27).

$$FU = fV - \frac{1}{\lambda}u(v)V,$$
$$FU = -\lambda V - \frac{1}{\lambda}(-1-\lambda^2)V = \frac{1}{\lambda}V,$$
$$FU = \frac{1}{\lambda}V,$$

which is equation (2.26).

$$\begin{aligned}(uoF)(X) &= (uof)(X) - \frac{1}{\lambda}u(X)u(V)\\ &= (uof)(X) - \frac{1}{\lambda}u(X)(-1-\lambda^2)\\ &= -\lambda u(X) + \frac{1}{\lambda}u(X) + \lambda u(X)\\ &= \frac{1}{\lambda}u(X),\end{aligned}$$
$$uoF = \frac{1}{\lambda}u,$$
$$FV = fV - \frac{1}{\lambda}u(V)V = fV = \lambda U,$$

which is equation (2.28) here equations (2.18), (2.19), (2.20), (2.21), (2.22), (2.23) are used. □

Lemma 2.4. Let M be a transversal hypersurface with hyperbolic (f, g, u, v, λ)-structure of an almost hyperbolic contact metric manifold $\overline{M}$. Then

(2.32)
$$\left(\overline{\nabla}_X \phi\right) Y = ((\nabla_X f) Y - u(Y) HX + h(X,Y) U) + \left(\left(\overline{\nabla}_X u\right) Y + h(X, fY) N\right)$$

(2.33)
$$\overline{\nabla}_X \xi = \nabla_X V - \lambda HX + (h(X,V) + X\lambda) N$$

(2.34) $$\left(\overline{\nabla}_X\phi\right) N = \left(-\nabla_X U + fHX\right)$$

(2.35) $$\left(\overline{\nabla}_X\eta\right) Y = \nabla_X u + \lambda h\left(X, Y\right),$$

for all $X, Y \in TM$. The proof is straight forward and hence omitted.

3. TRANSVERSAL HYPERSURFACES OF COSYMPLECTIC HYPERBOLIC MANIFOLD

Trans-Sasakian structures of type $(\alpha, 0)$ are called α-Sasakian and trans-Sasakian structures of type $(0, \beta)$ are called β-Kenmotsu structures. Trans-Sasakian structures of type $(0, 0)$ are called cosymplectic structures.

Theorem 3.1. Let M be a transversal hypersurfaces with hyperbolic (f, g, u, v, λ)-structure of a hyperbolic cosymplectic manifold $\overline{M}$. Then

(3.1) $$\left(\nabla_X f\right) Y = u\left(Y\right) HX - h\left(X, Y\right) U,$$

(3.2) $$\left(\nabla_X u\right) Y = -h\left(X, fY\right),$$

(3.3) $$\nabla_X V = \lambda HX,$$

(3.4) $$h\left(X, V\right) = -X\lambda,$$

(3.5) $$\nabla_X U = fHX,$$

(3.6) $$\left(\nabla_X v\right) = \lambda h\left(X, Y\right),$$

for all $X, Y \in TM$.

Proof. Using (1.4), (2.13), (2.15) in (2.32), we obtain

$$\left(\left(\nabla_X f\right) Y - u\left(Y\right) HX + h\left(X, Y\right) U\right) + \left(\left(\nabla_X u\right) Y + h\left(X, fY\right)\right) N = 0$$

Equating tangential and normal parts in the above equation, we get (3.1) and (3.2) respectively. Using (1.5) and (2.15) in (2.33), we have

$$\left(\nabla_X V - \lambda HX\right) + \left(h\left(X, V\right) + X\lambda\right) N = 0$$

Equating tangential and normal parts we get (3.3) and (3.4) respectively. Using (1.4), (2.14) and (2.15) in (2.34) and equating tangential, we get (3.5). In the last (3.6) follows from (2.35). □

Theorem 3.2. If M be a transversal hypersurface with hyperbolic (f, g, u, v, λ)-structure of a hyperbolic cosymplectic manifold, then the 2-form Φ on M is given by

$$\Phi(X, Y) \equiv g(X, fY)$$

is closed.

Proof. From (3.1) we get

$$(\nabla_X \Phi)(Y, Z) = h(X, Y)\, u(Z) - h(X, Z)\, u(Y),$$

which gives

$$(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y) = 0.$$

Hence the theorem is proved. □

Theorem 3.3. If M is a transversal hypersurface with almost product semi-Riemannian structure (F, G) of a hyperbolic cosymplectic manifold. Then the 2-form Ω on M is given by

$$\Omega(X, Y) = G(X, fY)$$

is closed.

Using (3.1), we calculate the Nijenhuis tensor

$$[F, F] = (\nabla_{FX} F)\, Y - (\nabla_{FY} F)\, X - F(\nabla_X F)\, Y + F(\nabla_Y F)\, X$$

and find that $[F, F] = 0$.

Therefore, in view of theorem (3.2), we have

Theorem 3.4. Every transversal hypersurface of a trans hyperbolic cosymplectic manifold, admits product structure.

4. TRANSVERSAL HYPERSURFACES OF TRANS HYPERBOLIC SASAKIAN MANIFOLDS

Theorem 4.1. Let M be a transversal hypersurface with hyperbolic (f, g, u, v, λ)–structure of a trans hyperbolic Sasakian manifold $\overline{M}$. Then

$$(4.1)\quad \begin{aligned}(\nabla_X f) Y &= \alpha (g(X,Y) V - v(Y) X) \\ &\quad +\beta (g(fX,Y) V - v(Y) fX) + u(Y) HX - h(X,Y) U\end{aligned}$$

$$(4.2)\quad (\nabla_X u) Y = \alpha\lambda g(X,Y) + \beta (\lambda g(fX,Y) - u(X) v(Y)) - h(X, fY).$$

$$(4.3)\quad \nabla_X V = \lambda HX - \alpha fX + \beta (X - v(X) V).$$

$$(4.4)\quad h(X,V) = \alpha u(X) - \beta\lambda v(X) - X\lambda.$$

$$(4.5)\quad \nabla_X U = fHX - \alpha\lambda X + \beta (\lambda fX - u(X) V).$$

$$(4.6)\quad (\nabla_X v) = \lambda h(X,Y) - \alpha g(fX,Y) + \beta (g(X,Y) - v(X) v(Y)),$$

for all $X, Y \in TM$.

Proof. Using (1.4), (2.13), (2.15) in (2.32), we obtain

$$\begin{aligned}&((\nabla_X f) Y - u(Y) HX + h(X,Y) U) + ((\nabla_X u) Y + h(X, fY)) N \\ = \;& \alpha (g(X,Y) V - v(Y) X) + \beta (g(fX,Y) V - v(Y) fX) \\ & +\alpha\lambda g(X,Y) + \beta\lambda g(fX,Y) - u(X) v(Y).\end{aligned}$$

Equating tangential and normal parts in the above equation, we get (4.1) and (4.2) respectively. Using (1.5) and (2.15) in (2.33), we have

$$\begin{aligned}&(\nabla_X V - \lambda HX) + (h(X,V) + X\lambda) N \\ = \;& -\alpha fX + \beta (X - v(X) V) - (\alpha u(X) + \beta\lambda v(X)) N.\end{aligned}$$

Equating tangential and normal parts we get (4.3) and (4.4) respectively. Using (1.4), (2.14) and (2.15) in (2.34) and equating tangential parts, we get (4.5) in the last (4.6) follows from (2.35). □

Theorem 4.2. If M be a transversal hypersurface with hyperbolic $(f, g, u, v, \lambda)-$ structure of a $(\alpha, 0)$ trans hyperbolic Sasakian manifold, then the 2-form Φ on M is given by

$$\Phi(X, Y) = g(X, fY)$$

is closed.

Proof. From (4.1) we get

$$\begin{aligned}(\nabla_X \Phi)(Y, Z) &= -\alpha(g(X, Y) v(Z) - g(X, Z) v(Y)) \\ &\quad -\beta(g(fX, Y) v(Z) - g(fX, Z) v(Y)) \\ &\quad +h(X, Y) u(Z) - h(X, Z) u(Y),\end{aligned}$$

which gives

$$\begin{aligned}&(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Z) \\ = {}& 2\beta(\Phi(X, Y) \eta(Z) + \Phi(Y, Z) \eta(X) + \Phi(Z, X) \eta(Y))\end{aligned}$$

If $\beta = 0$, then

$$(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Z) = 0,$$

that is

$$d\Phi = 0.$$

Hence the theorem is proved. □

Theorem 4.3. If M is a transversal hypersurface with almost product semi-Riemannian structure (F, G) of a $(\alpha, 0)$ trans hyperbolic Sasakian manifold. Then 2-form Ω on M is given by

$$\Omega(X, Y) = G(X, FY)$$

is closed.

Using (4.1), we calculate the Nijenhuis tensor

$$[F, F] = (\nabla_{FX} F) Y - (\nabla_{FY} F) X - F(\nabla_X F) Y + F(\nabla_Y F) X$$

and find that

$$[F, F] = 0$$

Therefore, in view of theorem 4.2, we have

Theorem 4.4. Every transversal hypersurface of a trans hyperbolic Sasakian manifold admits a product structure.

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Date received: September 9, 2007