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A CLASSICAL VIEWPOINT ON QUANTUM CHAOS

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ABSTRACT. The relationship between the Lie point symmetries of the Schrödinger equation – the so-called Schrödinger algebra – and the Noether point symmetries of the corresponding classical Lagrangian is proven. When the algebra of these point symmetries is sufficiently rich, the point symmetries can be used to construct the solutions of the Schrödinger equation. This is illustrated in the case of the simple harmonic oscillator and an Ermakov-Pinney system in $1+1$ dimensions. The presence of a sufficient number of Noether symmetries of the classical Lagrangian guarantees integrability. We propose the absence of these in the corresponding classical Lagrangian, where this exists, to be the cause of the phenomenon known as 'quantum chaos'.

1. INTRODUCTION

In the determination of the wave function of the Schrödinger equation for such problems as the simple harmonic oscillator and the Kepler-Coulomb problem the time and space variables are separated and use is made of operator techniques to construct the solution of the time-independent Schrödinger equation.

These operators are based on conserved quantities of the corresponding classical system, for example the angular momentum for a central force problem. The representation of the conserved quantity as an operator in position space is derived using the general rules given by Dirac following his identification of quantum mechanical operators with classical variables when the latter exist [2].

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The conserved quantities are first integrals and so are related to Noether symmetries according to

$$I = f - \left[\tau L + \eta_i - \dot{q}_i\tau\frac{\partial L}{\partial \dot{q}_i}\right], \tag{1}$$

where I is the first integral, L the classical Lagrangian, τ and η_i the coefficient functions of the Noether symmetry

$$G = \tau\frac{\partial}{\partial t} + \eta_i\frac{\partial}{\partial q_i} \tag{2}$$

and f the gauge function introducted into the derivation of the theorem [11] to allow for boundary terms due to the infinitesimal transformation in time produced by G. In the context of quantum mechanics (1) is better written in terms of the canonical variables (q_i, p_i) as [15, 9]

$$I = f + \tau H - \eta_i p_i. \tag{3}$$

Given a Lagrangian, the Noether symmetry and gauge function can be calculated from [14]

$$\dot{f} = \dot{\tau}L + \tau\frac{\partial L}{\partial t} + \eta_i\frac{\partial L}{\partial q_i} + (\dot{\eta}_i - \dot{q}_i\dot{\tau})\frac{\partial L}{\partial \dot{q}_i}. \tag{4}$$

The corresponding Schrödinger equation

$$\hat{H}u = i\frac{\partial u}{\partial t}, \tag{5}$$

when written in the position representation, will possess a Lie symmetry

$$G = \tau\frac{\partial}{\partial t} + \eta_i\frac{\partial}{\partial q_i} + \zeta\frac{\partial}{\partial u} \tag{6}$$

if

$$G^{[ext]}\left[\hat{H}u - i\frac{\partial u}{\partial t}\right]\Bigg|_{\hat{H}u - i\frac{\partial u}{\partial t}=0} = 0, \tag{7}$$

where $G^{[ext]}$ denotes the extension of (6) to include the derivatives present in (5). Normally the second extension would be used.

The Lie point symmetries of (5) are the elements of what is called the Schrödinger algebra of (5) [10]. Since the same algebra is obtained if (5) is replaced by a heat equation with the same space terms, there does not seem to be point to having a name other than the Lie algebra of the symmetries of the equation*.

In this paper we show that the classical Noether point symmetries of the Lagrangian and the Lie point symmetries of the corresponding Schrödinger equation are closely related in the case of a particle Lagrangian

$$L = \tfrac{1}{2}\dot{\mathbf{q}}.\dot{\mathbf{q}} - V(q,t). \tag{8}$$

In fact apart from the two Lie point symmetries generic to a linear partial differential equation a Lie point symmetry of (5) (for H derived from (8)) is a Noether point symmetry of (8). This means that the nongeneric Lie point symmetries of the Schrödinger equation are intimately connected with the classical first integrals derived from the Noether symmetries.

There are some classical problems which are rich in Noether symmetries. We use these symmetries (with the modification necessary to include the wave function u) as Lie symmetries of the Schrödinger equation to construct the wave functions of the time-dependent Schrödinger equation. The method is to construct a similarity solution of the Schrödinger equation and then to use the other symmetries to construct other solutions through the property that Lie symmetries map solutions into solutions [8]. In this respect we find that symmetries corresponding to nonautonomous classical symmetries and so the noninvariance algebra of the Hamiltonian are particularly valuable. We illustrate the method of construction of wave functions with the simple harmonic oscillator and the Ermakov-Pinney [3, 13] system both in one dimension.

Classically an absence of symmetry is associated with nonintegrabil-

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*A comprehensive and recent account of symmetries of the Schrödinger equation has been given by Zachary and Shtelen [18].

ity. More precisely it is the absence of a sufficient amount of symmetry of the right sort [12]. The definition of 'right sort' is a difficult task. To give a single example the Hénon-Heiles Hamiltonian displays chaotic behaviour except for a few pairs of values of the parameters. The symmetry, in addition to $\partial/\partial t$, in one of the integrable cases is quite nonlocal [12]. In terms of symmetry we can state that such and such a system is integrable because it possesses this set of symmetries of the right sort. Thus, if a scalar third order ordinary differential equation has three point symmetries, we know that it is integrable since the equation can be reduced to quadratures and this is one of the definitions of integrability. The evidence to prove chaos from the viewpoint of symmetry does not yet exist.

Claims of the existence of chaos in the solution of an evolution equation such as the Schrödinger equation have been beset by the obvious linearity of the equation. One must resort to irregularity in the distribution of energy levels or the pattern of the nodal lines of the wave function as criteria for 'quantum chaos'. Both of these features can be associated with nonseparability of the time-independent Schrödinger equation. We suggest that the underlying connection between classically chaotic systems and their quantum counterparts lies in the connection between the Noether symmetries of the classical Lagrangian and the Lie symmetries of the Schrödinger equation. In the quantum picture the lack of a sufficient number of first integrals due to a lack of a sufficient number of Noether symmetries in the classical problem is reflected in the irregularities found in the spectra and nodal lines which are a reflection of nonseparability.

We cannot yet offer a proof of the assertion because we, not to mention others, cannot give a proper definition of a symmetry of the 'right sort'. The resolution of this problem is one of the central problems left in the whole business of Lie and Noether symmetries.

2. NOETHER POINT SYMMETRIES FOR PARTICLE LARANGIANS

With the coefficient functions and f functions of t and q_i only and L as given by (8), (4) becomes

$$\begin{aligned}\frac{\partial f}{\partial t}+\dot{q}_i\frac{\partial f}{\partial q_i} &= \frac{\partial \tau}{\partial t}+\dot{q}_i\frac{\partial \tau}{\partial q_i}\tfrac{1}{2}\dot{q}_j\dot{q}_j - V - \tau\frac{\partial V}{\partial t}-\eta_i\frac{\partial V}{\partial q_i}\\ &+ \frac{\partial \eta_i}{\partial t}+\dot{q}_j\frac{\partial \eta_i}{\partial q_j}-\dot{q}_i\frac{\partial \tau}{\partial t}-\dot{q}_i\dot{q}_j\frac{\partial \tau}{\partial q_j}\dot{q}_i \qquad (9)\end{aligned}$$

in which summation over repeated indices is understood. Equation (9) gives the set of determining equations

$$\begin{aligned} 0 &= \tfrac{1}{2}\frac{\partial \tau}{\partial q_i}\\ 0 &= \frac{\partial \eta_i}{\partial q_j}+\frac{\partial \eta_j}{\partial q_i}-\delta_{ij}\frac{\partial \tau}{\partial t}\\ \frac{\partial f}{\partial q_i} &= -V\frac{\partial \tau}{\partial q_i}+\frac{\partial \eta_i}{\partial t}\\ \frac{\partial f}{\partial t} &= -V\frac{\partial \tau}{\partial t}-\tau\frac{\partial V}{\partial t}-\eta_i\frac{\partial V}{\partial q_i}. \qquad (10)\end{aligned}$$

From the first and second of (10)

$$\begin{aligned}\tau &= a(t)\\ \eta &= \tfrac{1}{2}\dot{a}\mathbf{q}+\mathbf{b}(t)\times\mathbf{q}+\mathbf{c}(t). \qquad (11)\end{aligned}$$

From the third of (10) the requirement that

$$\frac{\partial^2 f}{\partial q_i \partial q_j}=\frac{\partial^2 f}{\partial q_j \partial q_i} \qquad (12)$$

imposes the condition that $\mathbf{b}$ be a constant vector and consequently

$$f=\tfrac{1}{4}\ddot{a}r^2+\mathbf{c}.\mathbf{q}+d(t), \qquad (13)$$

where $r=|\mathbf{q}|$. The fourth of (10) is now the linear partial differential equation

$$\tfrac{1}{4}\dddot{a}\, r^2+\dot{\mathbf{c}}.\mathbf{q}+\dot{d}=-V\dot{a}-a\frac{\partial V}{\partial t}-\tfrac{1}{2}\dot{a}\mathbf{q}+\mathbf{b}\times\mathbf{q}+\mathbf{c}.\nabla V. \qquad (14)$$

In (14) the term $\mathbf{b} \times \mathbf{q}.\nabla V$ vanishes if the potential is due to a central force and the symmetries associated with angular momentum follow immediately.

For a given potential (14) gives the functions $a(t)$, $\mathbf{c}(t)$ and $d(t)$ and the constant $\mathbf{b}$. Naturally all functions $a(t)$, $\mathbf{c}(t)$ and $d(t)$ and constant $\mathbf{b}$ will be zero for an arbitrary given potential since one does not expect such potentials to have Noether symmetries or first integrals. If the potential is not given, (14) is a linear partial differential equation to determine $V(q,t)$ in terms of arbitrary functions $a(t)$, $\mathbf{c}(t)$ and $d(t)$ and arbitrary constant $\mathbf{b}$.

3. LIE POINT SYMMETRIES OF THE SCHRÖDINGER EQUATION

The time-dependent Schrödinger equation corresponding to the Lagrangian (8) is

$$\nabla^2 u - 2V(q,t)u + 2i\frac{\partial u}{\partial t} = 0. \tag{15}$$

We use Program `LIE` [5] to compute the determining equations which, after some initial internal simplification, are

$$\frac{\partial^2 f_4}{\partial u^2} = 0 \tag{16}$$

$$\frac{\partial f_\alpha}{\partial q_\beta} + \frac{\partial f_\beta}{\partial q_\alpha} = \delta_{\epsilon\beta}\dot{a} \tag{17}$$

$$-2i\frac{\partial f_\alpha}{\partial t} + 2\frac{\partial^2 f_4}{\partial u \partial q_\alpha} - \nabla^2 f_\alpha = 0 \tag{18}$$

$$\begin{aligned} 2i\frac{\partial f_4}{\partial t} + 2u\frac{\partial f_4}{\partial u}V - 2f_4 V - 2u\dot{a}V - 2u\frac{\partial V}{\partial t}a & \\ -2uf_\alpha\frac{\partial V}{\partial q_\alpha} + \nabla^2 f_4 &= 0 \end{aligned} \tag{19}$$

in which the Greek indices run from one to three and the symmetry is written as

$$G = a(t)\frac{\partial}{\partial t} + f_\alpha(t,q)\frac{\partial}{\partial q_\alpha} + f_4(t,q,u)\frac{\partial}{\partial u}. \tag{20}$$

From (16) and (17)

(21) $$f_4 = g_0(t,q) + ug_1(t,q)$$

(22) $$\mathbf{f} = \tfrac{1}{2}\dot{a}\mathbf{q} + \mathbf{b}(t)\times\mathbf{q} + \mathbf{c}(t).$$

From (18)

(23) $$g_1(t,q) = \tfrac{1}{4}i\ddot{a}r^2 + i\dot{\mathbf{c}}.\mathbf{q} + id(t)$$

and $\mathbf{b}(t)$ must be a constant. When (21) is substituted into (19), the terms independent of u give

(24) $$\nabla^2 g_0 - 2Vg_0 + 2i\frac{\partial g_0}{\partial t} = 0,$$

i.e. $g_0(t,q)$ is a solution of the original Schrödinger equation, and the coefficient of u gives

(25) $$\tfrac{1}{4}\,\ddot{a}\,q^2 + \dot{\mathbf{c}}.\mathbf{q} + \dot{d} - \tfrac{1}{2}i\ddot{a} = -\dot{a}V - a\frac{\partial V}{\partial t} - \tfrac{1}{2}\dot{a}\mathbf{q} + \mathbf{b}\times\mathbf{q} + \mathbf{c}.\nabla V.$$

If in (23) we rename the arbitrary function $d(t)$ through $\dot{d} = \dot{d}(t) + \frac{1}{2}i\ddot{a}$, (25) becomes identical to (14).

Thus we have the result that, if the particle Lagrangian (8) has the Noether point symmetry

(26) $$G = a(t)\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{a}\mathbf{q} + \mathbf{b}\times\mathbf{q} + \mathbf{c}(t).\nabla,$$

the corresponding Schrödinger equation has the Lie point symmetry

(27) $$G = a(t)\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{a}\mathbf{q} + \mathbf{b}\times\mathbf{q} + \mathbf{c}(t).\nabla + iu\tfrac{1}{4}\ddot{a}q^2 + \dot{\mathbf{c}}.\mathbf{q} + d + \tfrac{1}{2}i\dot{a}\frac{\partial}{\partial u}.$$

The Schřodinger equation also has the nonNoetherian Lie point symmetries

$$X_1 = g_0(t,q)\frac{\partial}{\partial u}$$

(28) $$X_2 = u\frac{\partial}{\partial u}.$$

The second of these is due to the definition of $d(t)$ in the expression for $g_1(t,q)$ through $\dot{d}$, i.e. up to an arbitrary constant.

4. WAVEFUNCTIONS FROM THE LIE POINT SYMMETRIES

In the case of a one-degree-of-freedom system to each symmetry of the form

$$G=\tau(t)\frac{\partial}{\partial t}+\eta(t,x)\frac{\partial}{\partial x}+u\zeta(t,x)\frac{\partial}{\partial u} \tag{29}$$

established as the structure of the Lie point symmetries for the Schrödinger equation (15) we associate a similarity solution by the following procedure. The similarity variables are found as the characteristics of G from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}x}{\eta}=\frac{\mathrm{d}u}{u\zeta} \tag{30}$$

and are

$$v=f(x,t)\qquad\text{and}\qquad w=ug(x,t). \tag{31}$$

We substitute

$$u=\frac{1}{g(t,x)}j(v) \tag{32}$$

into the Schrödinger equation (15) to obtain an ordinary differential equation for $j(v)$. The solution of this ordinary differential equation provides the similarity solution for u in terms of t and x through (31a) and (32).

The similarity solution defines a surface in (t,x,u) space through

$$\frac{j\circ f(t,x)}{ug(t,x)}=cst. \tag{33}$$

The left side of (33) is annihilated by G in (29). If there exists one or more other symmetries, their action on (33) is to transform solutions into solutions. Of the possible similarity solutions we reject those which are not square integrable. The rôle of G as an annihilator of the left hand side of (33) suggests that there may be another symmetry which plays the rôle of a creation operator and so lead to a whole class of solutions. If this be the case, any similarity solution which is not impossible becomes a candidate for the generation of realistic wave functions.

4.1. ONE DIMENSIONAL SIMPLE HARMONIC OSCILLATOR

LIE returns seven Lie point symmetries (the maximum for a 1 + 1 parabolic partial differential equation [7]) for the time-dependent Schrödinger equation

$$\frac{\partial^2 u}{\partial x^2} - x^2 u + 2i\frac{\partial u}{\partial t} = 0. \tag{34}$$

We write them as

$$\begin{aligned}
Y_1 &= e^{it}\frac{\partial}{\partial x} - ux\frac{\partial}{\partial u}\\
Y_2 &= e^{-it}\frac{\partial}{\partial x} + ux\frac{\partial}{\partial u}\\
Y_3 &= \frac{\partial}{\partial t}\\
Y_4 &= e^{2it}\left[\frac{\partial}{\partial t} + ix\frac{\partial}{\partial x} - iu(\tfrac{1}{2} + x^2)\frac{\partial}{\partial u}\right]\\
Y_5 &= e^{-2it}\left[\frac{\partial}{\partial t} + ix\frac{\partial}{\partial x} + iu(\tfrac{1}{2} - x^2)\frac{\partial}{\partial u}\right]\\
Y_6 &= u\frac{\partial}{\partial u}\\
Y_7 &= \phi(t,x)\frac{\partial}{\partial u},
\end{aligned} \tag{35}$$

where $\phi(t,x)$ is any solution of (34). The Noether point symmetries of

$$H = \tfrac{1}{2}p^2 + q^2, \tag{36}$$

the Hamiltonian corresponding to (34), and their associated first integrals are

$$\begin{aligned}
N_1 &= e^{it}\tfrac{\partial}{\partial x} & I_1 &= e^{it}(\dot{x} - ix)\\
N_2 &= e^{-it}\tfrac{\partial}{\partial x} & I_2 &= e^{-it}(\dot{x} + ix)\\
N_3 &= \tfrac{\partial}{\partial t} & I_3 &= \tfrac{1}{2}(\dot{x}^2 + x^2)\\
N_4 &= e^{2it}\tfrac{\partial}{\partial t} + ix\tfrac{\partial}{\partial x} & I_4 &= e^{2it}\left[\tfrac{1}{2}\dot{x}^2 - x^2 - ix\dot{x}\right]\\
N_5 &= e^{-2it}\tfrac{\partial}{\partial t} - ix\tfrac{\partial}{\partial x} & I_5 &= e^{-2it}\left[\tfrac{1}{2}\dot{x}^2 - x^2 + ix\dot{x}\right].
\end{aligned} \tag{37}$$

The Lie algebra of the Noether symmetries is $sl(2,R) \oplus_s 2A_1$ whereas that for $Y_1 - Y_6$ of the Lie symmetries is $\{sl(2,R) \oplus A_1\} \oplus_s 2A_1$, i.e. the Lie Brackets do not close without the inclusion of Y_6. Of the five Noetherian first integrals only I_3, the energy, is autonomous. The other four form a noninvariance algebra with I_3 under the operation of taking the Poisson Bracket ($\dot{x} = p$) provided that the identity, $\mathcal{I}$, is added which is suggestive of the need to include Y_6 with $Y_1 - Y_5$ to obtain closure.

We consider the similarity solutions associated with Y_1 and Y_4. The associated Lagrange's system for Y_1 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{-ux} \tag{38}$$

so that the similarity variables are

$$v = t \qquad \text{and} \qquad w = u \exp\left[\tfrac{1}{2}x^2\right]. \tag{39}$$

We substitute

$$u = j(v)\exp\left[-\tfrac{1}{2}x^2\right] \tag{40}$$

into (34) and find that $j(v)$ satisifies

$$2ij' - j = 0 \tag{41}$$

so that the similarity solution is

$$u(t,x) = \exp\left[-\tfrac{1}{2}it - \tfrac{1}{2}x^2\right]. \tag{42}$$

In the case of Y_4 the same procedure leads to the second order ordinary differential equation $j'' = 0$ and the similarity solution is

$$u = A_0 \exp\left[-\tfrac{1}{2}it - \tfrac{1}{2}x^2\right] + A_1 x \exp\left[-\tfrac{3}{2}it - \tfrac{1}{2}x^2\right]. \tag{43}$$

Since (34) is even in x, the solution set for u can always be chosen from a basis of functions which are either even or odd in x and so we write the solution as the two solutions

$$\begin{aligned} u_0 &= \exp\left[-\tfrac{1}{2}it - \tfrac{1}{2}x^2\right] \\ u_1 &= 2x\exp\left[-\tfrac{3}{2}it - \tfrac{1}{2}x^2\right], \end{aligned} \tag{44}$$

where the factor 2 has been included in u_1 from hindsight.

We have two similarity solutions. We denote their solution surfaces by

$$\sum_0 = u^{-1}u_0 = u^{-1}\exp\left[-\tfrac{1}{2}it - \tfrac{1}{2}x^2\right]$$

and

$$\sum_1 = u^{-1}u_1 = u^{-1}2x\exp\left[-\tfrac{3}{2}it - \tfrac{1}{2}x^2\right]. \tag{45}$$

The action of Y_2 on $\sum_0$ and $\sum_1$ gives

$$Y_2\sum_0 = -\sum_1 \qquad \text{and} \qquad Y_2\sum_1 = -\sum_2 \tag{46}$$

respectively, where $\sum_2 = u^{-1}(4x^2 - 2)\exp\left[-\tfrac{5}{2}it - \tfrac{1}{2}x^2\right]$.

The action of Y_5 on $\sum_0$ and $\sum_1$ gives

$$Y_5\sum_0 = \tfrac{1}{2}i\sum_2 \qquad \text{and} \qquad Y_5\sum_1 = \tfrac{1}{2}i\sum_3 \tag{47}$$

respectively, where $\sum_3 = u^{-1}(8x^3 - 12x)\exp\left[-\tfrac{7}{2}it - \tfrac{1}{2}x^2\right]$.

We see that Y_2 acts like the standard creation operator for the simple harmonic oscillator and that Y_5 acts as a double creation operator mapping even states into even states and odd states into odd states. The polynomial components we recognise as the first few of the Hermite polynomials. In general, if we define

$$\sum_n = u^{-1}H_n(x)\exp\left[-(n+\tfrac{1}{2})it - \tfrac{1}{2}x^2\right], \tag{48}$$

we find that

$$\begin{aligned} Y_2\sum_n &= -\sum_{n+1} & Y_5\sum_n &= \tfrac{1}{2}i\sum_{n+2} \\ Y_1\sum_n &= 2n\sum_{n-1} & Y_4\sum_n &= 2i(n-1)\sum_{n-2}. \end{aligned} \tag{49}$$

Y_1 and Y_4 act as annihilation operators in steps of one and two respectively.

In the usual scheme of things Y_3 would be of paramount importance due to the relationship with the Hamiltonian as the generator of time translations and the separation of variables in the time-dependent Schrödinger equation. Since

$$Y_3 \sum_n = -i(n + \tfrac{1}{2}) \sum_n, \tag{50}$$

$\sum_n$ is an eigenfunction of Y_3 and the usual identification would be more manifest if we wrote $Z_3 = iY_3$.

We observe that the wave functions for the one dimensional simple harmonic oscillator have been obtained with those Lie symmetries of the time-dependent Schrödinger equation which correspond to nonautonomous Noetherian integrals. The symmetries Y_4 and Y_5 produce the whole sequence of odd and even wave functions. They are asociated with the noninvariance first integrals, I_4 and I_5. Classically the counterparts to these integrals in two and three dimensions give the trajectory of the particle [6] whereas the autonomous quadratic integrals give only the orbit [4]. These symmetries are first order differential operators. In the conventional approach to the solution of the quantal oscillator second order operators are required to perform the same task for the time-independent wave functions.

4.2. THE ERMAKOV-PINNEY SYSTEM

The scalar Ermakov-Pinney equation [3, 13]

$$\ddot{x} + x = \frac{h}{x^3} \tag{51}$$

has, respectively, the Lagrangian, Hamiltonian and time-dependent Schrödinger equation

$$L = \tfrac{1}{2}\dot{x}^2 - x^2 - \frac{h}{x^2} \tag{52}$$

$$H = \tfrac{1}{2}p^2 + x^2 - \frac{h}{x^2} \tag{53}$$

$$\frac{\partial^2 u}{\partial x^2} - x^2 - \frac{h}{x^2}u + 2i\frac{\partial u}{\partial t} = 0. \tag{54}$$

Equation (54) has the five Lie point symmetries

$$
\begin{aligned}
Y_1 &= \psi(t,x)\frac{\partial}{\partial u}\\
Y_2 &= u\frac{\partial}{\partial u}\\
Y_3 &= \frac{\partial}{\partial t}\\
Y_4 &= e^{2it}\left[\frac{\partial}{\partial t} + ix\frac{\partial}{\partial x} - iu(\tfrac{1}{2}+x^2)\frac{\partial}{\partial u}\right]\\
Y_5 &= e^{-2it}\left[\frac{\partial}{\partial t} - ix\frac{\partial}{\partial x} + iu(\tfrac{1}{2}-x^2)\frac{\partial}{\partial u}\right].
\end{aligned}
\tag{55}
$$

The three Noether symmetries, Y_3, Y_4 and Y_5, have the Lie algebra $sl(2,R)$.

The associated Lagrange's system for Y_4, the only reasonable candidate for an annihilation operator, is

$$
\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{ix} = \frac{\mathrm{d}u}{-iu(\frac{1}{2}+x^2)}. \tag{56}
$$

The invariants are

$$
v = xe^{-it} \qquad \text{and} \qquad w = u\exp\left[\tfrac{1}{2}it + \tfrac{1}{2}x^2\right]. \tag{57}
$$

We put

$$
u = \exp\left[\tfrac{1}{2}it + \tfrac{1}{2}x^2\right] f(v) \tag{58}
$$

and, on substitution into (54), find that

$$
f(v) = K_1 v^{(1+\alpha)/2} + K_2 v^{(1-\alpha)/2}, \tag{59}
$$

where $\alpha = \sqrt{1+4h}$, provided $h > -\frac{1}{4}$.

For $h \geq 0$ the second solution is rejected since it is divergent at the origin and we have

$$
u_0(t,x) = xe^{-it(1+\alpha)/2}\exp\left[-\tfrac{1}{2}it + x^2\right]. \tag{60}
$$

We define the solution surface in (t, x, u) space by

$$\sum_0 = u^{-1}x^{(1+\alpha)/2}\exp\left[-\tfrac{1}{2}2 + \alpha it - \tfrac{1}{2}x^2\right]. \tag{61}$$

The action of Y_5 on the solution surface (61) leads to the new solution

$$u_1(t, x) = 2x^2 - 2 - \alpha xe^{-it(1+\alpha)/2}\exp\left[-\tfrac{1}{2}6 + \alpha it - \tfrac{1}{2}x^2\right]. \tag{62}$$

The polynomials generated are the generalised Laguerre polynomials, $L_n^{\alpha/2}(x^2)$ [1, 781]. The symmetry Y_5 can be used as a 'creation' operator and the symmetry Y_4 as an annihilation operator.

When $-\frac{1}{4} \leq h \leq 0$, both solutions in (59) are viable and we have

$$u_0^-(t, x) = xe^{-it(1-\alpha)/2}\exp\left[-\tfrac{1}{2}it + x^2\right] \tag{63}$$

$$u_0^+(t, x) = xe^{-it(1+\alpha)/2}\exp\left[-\tfrac{1}{2}it + x^2\right]. \tag{64}$$

Y_5 acts on both of (63) and (64) as a creation operator and Y_4 as an annihilation operator on the higher solutions. The two series of solutions are not related. However, the energy levels are interlaced.

5. INTEGRABILITY AND CHAOS

We have shown that the Lie point symmetries of the time-dependent Schrödinger equation corresponding to a particle Lagrangian are, apart from the two generic symmetries, equivalent to the Noether point symmetries of that Lagrangian. To each Noether symmetry there is a first integral of the Euler-Lagrange equation of the system. The existence of independent first integrals equal in number to the number of degrees of freedom of the source problem guarantees integrability of the classical problem by Liouville's Theorem [17].

In the two examples treated the Lie point symmetries of the time-dependent Schrödinger equation corresponding to the classical Noether symmetries provide the means to obtain the wave functions from the ground state wave function, a similarity solution from one of the symmetries, by means of the characteristic property of Lie symmetries of mapping solutions into solutions. A study of the three dimensional oscillator shows that one cannot expect the chosen symmetries to map eigenstates into eigenstates. Rather thay are mapped into combinations of eigenstates.

The existence of certain symmetries permits the separation of variables. The most obvious of these is $\partial/\partial t$ which allows one to go to the time-independent Schrödinger equation. Generally speaking this separation is insufficient to obtain the wave functions and there is a need of further symmetries to construct the wave functions. In the examples considered here the symmetries corresponding to the explicitly time-dependent classical integrals are those which act as ladder operators and permit the construction of the eigenfunctions. These classical integrals have their quantal counterparts which, since they do not have zero commutator with the Hamiltonian operator, are elements of the noninvariance algebra. It would appear that the existence of such integrals is intimately connected with 'nice' solutions to the Schrödinger equation.

In systems of more than one degree-of-freedom one finds the phenomenon known as quantum chaos. These systems are characterised by an irregularity in the nodal lines of the wave function or the distribution of energy levels [16, 246*ff*]. We believe that the absence of Lie point symmetries of the time-dependent Schrödinger equation is related to this phenomenon just as the lack of Noether symmetries, and so first integrals, is a feature of classically chaotic systems.

In this paper we have highlighted the duality of the Lie point symmetries of the time-dependent Schrödinger equation and Noether point symmetries of the classical Lagrangian when both exist. However, this is only part of the story. In Classical Mechanics point symmetries are but a small part of the whole symmetry scene. There are also generalised symmetries, which contain derivatives of the dependent variables, and nonlocal symmetries, which contain integrals of the dependent variables. The use of the former is well established in the context of Noether's Theorem. The Lagrangian itself is not unique. These additional aspects must be considered thoroughly before the discussion of the symmetry basis of quantum chaos is complete.

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