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ON SUBMANIFOLDS OF CODIMENSION TWO OF A NEARLY TRANS-SASAKIAN MANIFOLD

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ABSTRACT. Some sufficient conditions for the submanifold of codimension two of a nearly trans-Sasakian manifold, with trivial normal bundle, to admit a nearly trans-Sasakian structure are obtained.

1. INTRODUCTION

In [12], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: **(1)** homogeneous normal contact Riemannian manifolds with $c > 0$, **(2)** global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and **(3)** a warped product space $\mathbb{R}\times_f\mathbb{C}^n$ if $c < 0$. It is known that the manifolds of class **(1)** are characterized by admitting a Sasakian structure. The manifolds of class **(2)** are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu ([8]) characterized the differential geometric properties of the manifolds of class **(3)**; the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian ([8]).

In the Gray-Hervella classification of almost Hermitian manifolds ([5]), there appears a class, $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold $\bar{M}$ is called a *trans-Sasakian structure* ([11]) if the product manifold

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$\bar{M} \times \mathbb{R}$ belongs to the class $\mathcal{W}_4$. The class $\mathcal{C}_6 \oplus \mathcal{C}_5$ ([10]) coincides with the class of trans-Sasakian structures of type (α, β). We note that trans-Sasakian structures of type $(0,0)$ are cosymplectic ([1]), trans-Sasakian structures of type $(0, \beta)$ are β-Kenmotsu ([6]) and trans-Sasakian structures of type $(\alpha, 0)$ are α-Sasakian ([6]).

Recently, C. Gherghe ([4]) introduced a nearly trans-Sasakian structure of type (α, β), which generalizes trans-Sasakian structure in the same sense as nearly Sasakian structures generalize Sasakian ones. A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly-Sasakian ([3]) or nearly Kenmotsu ([13]) or nearly cosymplectic ([1]) according as $\beta = 0$ or $\alpha = 0$ or $\alpha = 0 = \beta$.

In the present paper, we study the submanifolds of codimension two of nearly trans-Sasakian manifolds. Section 2 contains necessary details about nearly trans-Sasakian structures and a submanifold of codimension two of an almost contact metric manifold. Definition of nearly trans-Sasakian manifolds is given. It is shown that a submanifold of codimension two of an almost contact metric manifold admits a $\left(f, g, U_{(k)}, u_{(k)}, a_{(k)}\right)$ structure ([9], [14]) and under certain conditions this structure yields an almost contact metric structure on the submanifold ([7]). In the last section 3, in the main theorem of the paper, we establish that under certain conditions the induced almost contact metric structure on a submanifold of codimension two of a nearly trans-Sasakian manifold is also nearly trans-Sasakian. Finally, similar results are stated for some particular types of almost contact metric structures which are nearly trans-Sasakian: nearly Sasakian, nearly Kenmotsu, nearly cosymplectic, trans-Sasakian, Kenmotsu and cosymplectic.

2. NEARLY TRANS-SASAKIAN MANIFOLDS

Let $\bar{M}$ be an almost contact metric manifold ([1]) with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a vector field; η is 1-form and g is a compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \tag{3}$$

for all $X, Y \in T\bar{M}$.

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a *trans-Sasakian structure* (Oubina, [11]) if $(\bar{M} \times \mathbb{R}, J, G)$ belongs to the class $\mathcal{W}_4$ ([5]), where J is the almost complex structure on $\bar{M} \times \mathbb{R}$ defined by

$$J(X, ad/dt) = (\phi X - a\xi, \eta(X)d/dt)$$

for all vector fields X on $\bar{M}$ and smooth functions a on $\bar{M} \times \mathbb{R}$ and G is the product metric on $\bar{M} \times \mathbb{R}$. This may be expressed by the condition (Blair and Oubina [2])

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{4}$$

for some smooth functions α and β on $\bar{M}$, and we say that the trans-Sasakian structure is of type (α, β).

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a *nearly trans-Sasakian structure* ([4]) if

$$\begin{aligned} (\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X &= \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) \\ &\quad -\beta(\eta(Y)\phi X + \eta(X)\phi Y). \end{aligned} \tag{5}$$

A trans-Sasakian structure is always nearly trans-Sasakian.

Let M be a submanifold of codimension two of an almost contact metric manifold $\bar{M}$ endowed with an almost contact metric structure (ϕ, ξ, η, g). We identify M with $i(M)$ in $\bar{M}$, where $i : M \to \bar{M}$ is the imbedding map. We assume that there is a normal frame $\{N_1, N_2\}$ defined globally over the submanifold M (that is, the normal plane bundle is trivial). Then we have the decomposition law for ξ, $\phi i_* X$ and ϕN_r $(r = 1, 2)$ with respect to the frame $\{N_1, N_2\}$ as follows:

$$\xi = i_* U_3 - a_2 N_1 - a_1 N_2, \tag{6}$$

$$\phi i_* X = i_* f X + u_1(X) N_1 + u_2(X) N_2, \tag{7}$$

$$\phi N_1 = -i_* U_1 + a_3 N_2, \tag{8}$$

$$\phi N_2 = -i_* U_2 - a_3 N_1, \tag{9}$$

where f denotes a $(1,1)$ tensor field, U_r, $r = 1,2,3$ three vector fields, u_r, $r = 1,2$, two 1-forms, a_r, $r = 1,2,3$, three scalar fields on M. The induced Riemannian metric g' on M is given by

$$g'(X,Y) = g(X,Y), \qquad X,Y \in TM. \tag{10}$$

We define a 1-form u_3 on M by

$$u_3(X) = \eta(i_* X), \qquad X,Y \in TM. \tag{11}$$

Then one obtains the following equations :

$$f^2 = -I + u_1 \otimes U_1 + u_2 \otimes U_2 + u_3 \otimes U_3, \tag{12}$$

$$\left.\begin{array}{ll} fU_1 = -a_3U_2 + a_2U_3, & u_1 \circ f = a_3u_2 - a_2u_3, \\ fU_2 = a_1U_3 + a_3U_1, & u_2 \circ f = -a_1u_3 - a_3u_1, \\ fU_3 = -a_2U_1 - a_1U_2, & u_3 \circ f = a_2u_1 + a_1u_2, \end{array}\right\} \tag{13}$$

$$\left.\begin{array}{c} g'(X,fY) = -g'(fX,Y), \\ g'(X,U_i) = u_i(X), \\ g'(fX,fY) = g'(X,Y) - \sum_i u_i(X)\, u_i(Y) \end{array}\right\} \tag{14}$$

$$\left.\begin{array}{l} g'(U_1,U_1) = |U_1|^2 = 1 - a_2^2 - a_3^2, \\ \qquad g'(U_1,U_2) = u_1(U_2) = u_2(U_1) = -a_1a_2, \\ g'(U_2,U_2) = |U_2|^2 = 1 - a_3^2 - a_1^2, \\ \qquad g'(U_2,U_3) = u_2(U_3) = u_3(U_2) = a_2a_3, \\ g'(U_3,U_3) = |U_3|^2 = 1 - a_1^2 - a_2^2, \\ \qquad g'(U_3,U_1) = u_3(U_1) = u_1(U_3) = -a_3a_1. \end{array}\right\} \tag{15}$$

Thus a submanifold of codimension two of an almost contact metric manifold admits a $(f, g, U_{(k)}, u_{(k)}, a_{(k)})$ structure given in [9] (See also [14]). If $\varkappa$ is a scalar field defined by $\varkappa = \sqrt{a_2^2 + a_1^2 + a_3^2}$ on the submanifold of codimension two of an almost contact metric manifold, then $\varkappa$ is independent of a choice of the normal frame $\{N_1, N_2\}$ and satisfies $0 \leq \varkappa \leq 1$ ([7]).

Defining

$$U = a_3U_3 - a_1U_1 + a_2U_2, \qquad u = a_3u_3 - a_1u_1 + a_2u_2, \tag{16}$$

in view of (12)-(15) it follows that

$$\left.\begin{array}{c} u(U) = \varkappa^2, \quad u(X) = g'(X,U), \quad fU = 0, \quad u \circ f = 0, \\ f^3X + fX = u_1(fX)\, fU_1 + u_2(fX)\, fU_2 + u_3(fX)\, fU_3, \\ f^4X + (\varkappa^2 + 1)\, f^2X + \varkappa^2 X = u(X)\, U, \\ f^5X + (\varkappa^2 + 1)\, f^3X + \varkappa^2 fX = 0 \end{array}\right\} \tag{17}$$

for all $X \in TM$.

For a submanifold of codimension two of an almost contact metric manifold to admit an almost contact metric structure, it is sufficient that a condition ([7])

$$\text{either} \quad \varkappa = 0 \quad \text{on } M \qquad \text{or} \quad 0 < \varkappa \leq 1 \quad \text{on } M.$$

Thus if the scalar field $\varkappa$ $(0 \leq \varkappa \leq 1)$ is constant on M, then an almost contact metric structure can always be induced on M. In case of $0 < \varkappa \leq 1$, defining

$$(18)\quad f' = \frac{1}{\varkappa(\varkappa+1)}\left(f^3 + \left(\varkappa^2 + \varkappa + 1\right) f\right), \qquad u' = \frac{1}{\varkappa}u, \qquad U' = \frac{1}{\varkappa}U$$

one can verify that (f', u', U', g') is an almost contact metric structure on M.

3. SUBMANIFOLDS OF CODIMENSION TWO OF NEARLY TRANS-SASAKIAN MANIFOLDS

For the submanifold M of codimension two of an almost contact metric manifold, we have Gauss and Weingarten formulae as

$$(19)\qquad \bar{\nabla}_{i_*X} i_*Y = i_*\nabla_X Y + h_1(X,Y)\,N_1 + h_2(X,Y)\,N_2,$$

$$(20)\quad \bar{\nabla}_{i_*X} N_1 = -i_*H_1X + w(X)\,N_2, \qquad \bar{\nabla}_{i_*X} N_2 = -i_*H_2X - w(X)\,N_1,$$

where h_1 and h_2 (resp. H_1 and H_2) are the second fundamental tensor fields of type $(0,2)$ (resp. $(1,1)$) and w is the third fundamental 1-form with respect to $\{N_1, N_2\}$. h_1 and h_2 are symmetric and satisfy

$$h_1(X,Y) = g'(H_1X, Y), \qquad h_2(X,Y) = g'(H_2X, Y).$$

Now, we assume that $\bar{M}$ is nearly trans-Sasakian; and then for all $X, Y \in TM$, we have

$$\begin{aligned}(\bar{\nabla}_{i_*X}\phi)i_*Y + (\bar{\nabla}_{i_*Y}\phi)i_*X &= \alpha(2g(i_*X, i_*Y)\xi - \eta(i_*Y)i_*X - \eta(i_*X)i_*Y) \\ (21)\qquad\qquad &\quad -\beta(\eta(i_*Y)\phi i_*X + \eta(i_*X)\phi i_*Y).\end{aligned}$$

By using (7)-(9), (19) and (20), the left hand side of (21) reduces to

$$i_*\left((\nabla_X f)\,Y - u_1(Y)\,H_1X - u_2(Y)\,H_2X + h_1(X,Y)\,U_1 + h_2(X,Y)\,U_2\right)$$

$$
\begin{aligned}
&+i_*\left(\left(\nabla_Y f\right) X - u_1(X) H_1 Y - u_2(X) H_2 Y + h_1(X,Y) U_1 + h_2(X,Y) U_2\right)\\
&+\left(h_1(X,fY) + \left(\nabla_X u_1\right) Y - w(X) u_2(Y) + a_3 h_2(X,Y)\right) N_1\\
&+\left(h_1(Y,fX) + \left(\nabla_Y u_1\right) X - w(Y) u_2(X) + a_3 h_2(X,Y)\right) N_1\\
&+\left(h_2(X,fY) + \left(\nabla_X u_2\right) Y + w(X) u_1(Y) - a_3 h_1(X,Y)\right) N_2\\
&+\left(h_2(Y,fX) + \left(\nabla_Y u_2\right) X + w(Y) u_1(X) - a_3 h_1(X,Y)\right) N_2;
\end{aligned}
$$

and using (6), (7), (10) and (11) the right hand side of (21) reduces to

$$
\begin{aligned}
&i_*\left(\alpha\left(2g'(X,Y)U_3 - u_3(Y)X - u_3(X)Y\right) - \beta\left(u_3(Y)fX + u_3(X)fY\right)\right)\\
&-\left(2\alpha a_2 g'(X,Y) + \beta u_3(Y) u_1(X) + \beta u_3(X) u_1(Y)\right) N_1\\
&-\left(2\alpha a_1 g'(X,Y) + \beta u_3(Y) u_2(X) + \beta u_3(X) u_2(Y)\right) N_2.
\end{aligned}
$$

Comparing the tangential parts to M in the above two expressions, we obtain

$$
\begin{aligned}
&\left(\nabla_X f\right) Y + \left(\nabla_Y f\right) X\\
=\ &\alpha\left(2g'(X,Y)U_3 - u_3(Y)X - u_3(X)Y\right) - \beta\left(u_3(Y)fX + u_3(X)fY\right)\\
&+u_1(Y) H_1 X + u_2(Y) H_2 X + u_1(X) H_1 Y + u_2(X) H_2 Y\\
&-2\left(g'(H_1 X,Y) U_1 + g'(H_2 X,Y) U_2\right) \qquad (22)
\end{aligned}
$$

for all $X, Y \in TM$.

Now, we present the main theorem.

Theorem 3.1 *Suppose that $\varkappa = 1$ on a submanifold M of codimension two of a nearly trans-Sasakian manifold $\bar{M}$ with trivial normal plane bundle. Then the induced almost contact metric structure on M will be nearly trans-Sasakian if the second fundamental tensors H_1 and H_2 are related by $a_1 H_1 = a_2 H_2$.*

Proof. Since $1 = \varkappa^2 = {a_2}^2 + {a_1}^2 + {a_3}^2$, therefore in view of Lemma 1.1 of [9], U_1, U_2 and U_3 are linearly dependent. Therefore, we have

$$
|U_1| \cdot |U_2| = g'(U_1, U_2), \quad |U_2| \cdot |U_3| = g'(U_2, U_3), \quad |U_3| \cdot |U_1| = g'(U_3, U_1).
$$

Since $\varkappa = 1$ from (18) we have $U = U'$ and $u = u'$. Then it can be easily verified that

$$
U_1 = -a_1 U, \quad U_2 = a_2 U \quad \text{and} \quad U_3 = a_3 U,
$$

$$
u_1 = -a_1 u, \quad u_2 = a_2 u, \quad u_3 = a_3 u.
$$

Also, from (17) and (18) it follows that $f' = f$.

Hence, (22) reduces to

$$\begin{aligned}(\nabla_X f)\,Y + (\nabla_Y f)\,X \;&=\; a_3\alpha\left(2g'(X,Y)U - u(Y)X - u(X)Y\right)\\ &\quad -a_3\beta\left(u(Y)fX + u(X)fY\right)\\ &\quad -u\,(Y)\,(a_1H_1 - a_2H_2)\,X - u\,(X)\,(a_1H_1 - a_2H_2)\,Y\\ &\quad -2g'\left((a_1H_1 - a_2H_2)\,X, Y\right)U\end{aligned}$$

for all $X, Y \in TM$. From above equation it follows that the almost contact metric structure (f, u, U, g') on M will be nearly trans-Sasakian if $a_1H_1 = a_2H_2$.

In view of the above theorem, we are able to state the following theorem.

Theorem 3.2 *Suppose that $\varkappa = 1$ on a submanifold M of codimension two of an almost contact metric manifold $\bar{M}$ with trivial normal plane bundle. Then the induced almost contact metric structure on M will be nearly Sasakian, nearly Kenmotsu, nearly cosymplectic, trans-Sasakian,Sasakian, Kenmotsu and cosymplectic according as the almost contact metric structure on $\bar{M}$ is nearly Sasakian, nearly Kenmotsu, nearly cosymplectic,trans-Sasakian, Sasakian, Kenmotsu and cosymplectic respectively, provided the second fundamental tensors H_1 and H_2 satisfy $a_1H_1 = a_2H_2$.*

A submanifold M is called ϕ-invariant if $\phi i_*X \in TM$ for all $X \in TM$ or equivalently, if $\phi i_*X = i_*fX$ holds identically. Hence, $a_1 = 0 = a_2$ and ${a_3}^2 = 1$. Accordingly, we have $\varkappa = 1$. Thus we are able to state the following

Theorem 3.3 *Suppose that M is a ϕ-invariant submanifold of codimension two of an almost contact metric manifold $\bar{M}$ with trivial normal plane bundle. Then the induced almost contact metric structure on M will be nearly trans-Sasakian, nearly Sasakian, nearly Kenmotsu, nearly cosymplectic, trans-Sasakian, Sasakian, Kenmotsu and cosymplectic according as the almost contact metric structure on $\bar{M}$ is nearly trans-Sasakian, nearly Sasakian, nearly Kenmotsu, nearly cosymplectic, trans-Sasakian, Sasakian, Kenmotsu and cosymplectic respectively.*

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