Arab J. Math. Sc.
Volume 9, Number 1, June 2003
pp. 1-14

A Model For Fuzzy Plane Projective Geometry

M. Tahami, Y. Bahrampour and M. Mashinchi

ABSTRACT. In this note we show that a recently defined projective geometry on a given three dimensional fuzzy vector space is a model for the fuzzy projective geometry presented by Gupta and Ray (1993). Moreover we show that the fuzzy Desargues' proposition is valid in this model.

1. INTRODUCTION

Ever since the publication of Zadeh's classic paper on fuzzy sets [21], which introduced a gradual method for pattern differentiation, an approach qualitatively different from stochastic one, fuzzy techniques have been applied with startling generalized effect in almost all branches of mathematics. As far as it is known to the authors, in the field of geometry, the primary effort for fuzzification of geometry was carried out by A. Rosenfeld [15]. Since then many other papers [1, 15-19] have appeared in this field. K. C. Gupta and S. Ray [5] have studied the concept of a fuzzy plane projective geometry FPPG using fuzzy points as the fundamental notion. The construction of a fuzzy projective geometry FPG(V), from V, where V is either a fuzzy vector space or a fuzzy group is studied by L. Kuijken et al. [7,8]. The excellent book by J. N. Mordeson and P. S. Nair [10], reviews fuzzy geometry, among other concepts. More recent works are reported in [2-3, 11-13]. In this note, using the results in [5] and [7,8], in the case where V is a fuzzy three dimensional vector space, we show that FPG(V) is a model for FPPG. Moreover it is proved that the fuzzy Desargues' proposition is valid in the model FPG(V).

2. PRELIMINARIES

In this section we review some concepts needed in subsequent sections. For more details see the references.

Definition 2.1. [5] Let S be a nonempty set, $\mathbf{x} \in S$ and $\lambda \in (0,1]$. A fuzzy point $\mathbf{x} = (\mathbf{x}, \lambda)$ of S is the fuzzy subset of S that maps $\mathbf{x}$ to λ and other points to 0.

Definition 2.2. [5] Let S be a nonempty set.
1) A collection π of fuzzy points of S is called a complete set of fuzzy points if, given $\mathbf{x} \in S$, there exists $\lambda \in (0,1]$ such that $(\mathbf{x}, \lambda) \in \pi$.
2) Two fuzzy points $(\mathbf{x}, \alpha), (\mathbf{y}, \beta) \in \pi$ are said to be fuzzy distinct if $\mathbf{x} \neq \mathbf{y}$. If $\mathbf{x} = \mathbf{y}$ and $\alpha \neq \beta$, they are said to be fuzzy vertical.
3) A nonzero fuzzy subset L of S is called a fuzzy line through π, if $L(\mathbf{x}) > 0$ implies $\mathbf{x} = (\mathbf{x}, L(\mathbf{x})) \in \pi$, for all $\mathbf{x} \in S$.
4) A fuzzy line L is incident with the fuzzy point $(\mathbf{x}, \lambda) \in \pi$, if $L(\mathbf{x}) = \lambda$. In this case we write $(\mathbf{x}, \lambda) IL$ or $\mathbf{x} \in L$, and we call I the incident relation. We some times say that the fuzzy point $(\mathbf{x}, \lambda)$ lies on the fuzzy line L or the fuzzy point $(\mathbf{x}, \lambda)$ belongs to the fuzzy line L.
5) Two or more fuzzy points $(\mathbf{x}_i, \alpha_i)$, $i = 1, 2, \ldots .n$, are fuzzy colinear if there is a fuzzy line with which each of them is incident.
6) Two fuzzy lines L and M are distinct if there is a fuzzy point $(\mathbf{x}, \lambda)$ such that L is incident with $(\mathbf{x}, \lambda)$ and M is not, or M is incident with $(\mathbf{x}, \lambda)$ and L is not.

2.1 Fuzzy point approach of Gupta and Ray

In [5], an axiomatic approach for the fuzzification of a plane projective geometry is discussed. Below we give a short review of this approach.

A fuzzy plane projective geometry FPPG is an axiomatic theory with the triple (π, Λ, I) as its fundamental notion, where π is a complete set of fuzzy points of a non-empty set S, Λ is a collection of fuzzy lines through π, and I is an incident relation which satisfies the following three axioms:

F_1: Given two distinct points in π, there exists a unique fuzzy line in Λ with which both are incident.
F_2: Given two distinct fuzzy lines in Λ there is at least one fuzzy point in π with which both are incident.
F_3: π contains at least four fuzzy distinct points such that no three of them are incident with one and the same fuzzy line in Λ.

Definition 2.1.1. [5]. A fuzzy triangle $(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3, L_1L_2L_3)$ is a set of three fuzzy distinct points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and a set of three fuzzy lines L_1, L_2, L_3 such that $\mathbf{x}_i$ belongs to L_k for all $i \neq k$, and $\mathbf{x}_i \notin L_i$. The points $\mathbf{x}_i$ are the vertices and the line L_i are the sides. For two fuzzy triangles $(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3, L_1L_2L_3)$ and $(\mathbf{x}'_1\mathbf{x}'_2\mathbf{x}'_3, L'_1L'_2L'_3)$, $\mathbf{x}_i$ and $\mathbf{x}'_i$ are corresponding vertices, L_i and L'_i are corresponding sides.

2.2 Fuzzy Desargues' Proposition [5]

Let two fuzzy triangles $(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3, L_1L_2L_3)$ and $(\mathbf{x}'_1\mathbf{x}'_2\mathbf{x}'_3, L'_1L'_2L'_3)$ be given. If every two corresponding vertices are fuzzy distinct, every two corresponding sides are distinct and the fuzzy lines connecting corresponding vertices are incident with a fuzzy point O, then the corresponding sides intersect in three fuzzy points which are either fuzzy colinear or fuzzy vertical.

2.3 Fuzzy vector space

Definition 2.3.1. [7]. Let V be a finite n-dimensional vector space over a field K and $\lambda : V \to [0,1]$ be a fuzzy set on V. Then we call λ a fuzzy n-dimensional vector space on V if and only if $\lambda(a\mathbf{x} + b\mathbf{y}) \geq \min(\lambda(\mathbf{x}), \lambda(\mathbf{y}))$ for all $a, b \in K$ and for all $\mathbf{x}, \mathbf{y} \in V$. If λ is a fuzzy vector space on V, we write (V, λ).

It is clear that if (V, λ) is a fuzzy n-dimensional vector space and U is any subspace of V, then $(U, \lambda|U)$ is also a fuzzy p-dimensional vector space, where $p = \dim(U)$.

In the classical case, for a given vector space V, a projective space $PG(V)$ is defined [6] to be the pair $(D(V), I)$, where $D(V)$ is a collection of subspaces of V with the relation I on $D(V)$ such that U_1 and U_2 are I-related if either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. A fuzzification of this classical notion is as follows: Given a fuzzy vector space V, a fuzzy projective space $FPG(V)$ is defined to be the pair $(FD(V), FI)$, where $FD(V)$ is a collection of fuzzy vector subspaces of an n-dimensional fuzzy vector space (V, λ) with FI relation, such that

$$(U_1, \lambda_1)\, FI\, (U_2, \lambda_2) \text{ if and only if } U_1 \subseteq U_2 \text{ or } U_2 \subseteq U_1.$$

One can see that the relation I induces an FI relation and vice-versa.

Definition 2.3.2. [7]. Suppose V is an n-dimensional vector space. A flag of length m in V is a sequence of distinct, non-trivial subspaces $(U_0, U_1, ..., U_m)$ such that $U_j \subseteq U_i$ for all $j > i \geq n-1$. The rank of a flag is the number of subspaces it contains. A maximal flag in V is a flag of length n.

We need the following theorem from [7] to prove our main results.

Theorem 2.3.3. [7]. If $\lambda : V \longrightarrow [0,1]$ is a 3-dimensional fuzzy vector space on V, then there exists a (not necessarily unique) maximal flag (U_0, U_1, U_2, V) of length 3, where $\dim(U_i) = i$ and four real numbers $a_0 \geq a_1 \geq a_2 \geq a_3$ in $[0,1]$ such that λ is of following form:

$$\lambda : V \longrightarrow [0, I]$$

$$\begin{array}{lcl} u \mapsto a_0 & \text{for} & u \in \{0\} = U_0 \\ u \mapsto a_1 & \text{for all} & u \in U_1 - U_0 \\ u \mapsto a_2 & \text{for all} & u \in U_2 - U_1 \\ u \mapsto a_3 & \text{for all} & u \in V - U_2. \end{array}$$

3. MAIN RESULTS

Let (V, λ) be a 3-dimensional vector space over the field K, and let $FPG(V)$ be the collection of all the fuzzy subspaces of dimensions 1 and 2. Our aim is to convert $FPG(V)$ into a model of $FPPG$, for which we regard the 1-dimensional fuzzy subspaces as the points and the 2-dimensional subspaces as the lines.

Theorem 3.1. If (V, λ) is a 3-dimensional fuzzy vector space over the field K, then $FPG(V)$ is a model of $FPPG$, as defined in [5].

Proof. By Theorem 2.3.3, λ is defined by

$$\lambda : V \longrightarrow [0, I]$$

$$\begin{array}{lll} u \longmapsto a_0 & \text{for} & u \in \{0\} = U_0 \\ u \longmapsto a_1 & \text{for all} & u \in U_1 - U_0 \\ u \longmapsto a_2 & \text{for all} & u \in U_2 - U_1 \\ u \longmapsto a_3 & \text{for all} & u \in V - U_2. \end{array}$$

Let $V = \langle e_1, e_2, e_3 \rangle$. Without loss of generality assume $U_1 = \langle e_1 \rangle$ and $U_2 = \langle e_1, e_2 \rangle$. Take $FD(V)$ as all the 1-dimensional and 2-dimensional fuzzy vector subspaces of (V, λ). Now, let S be the set of all 1-dimensional subspaces of V over K. Then the set π of fuzzy points and the set Λ of fuzzy lines for the $FPG(V)$ model are defined as follows:
Let

$$\begin{aligned} \pi &= \{(U, \lambda) | U \in S, (U, \lambda) \text{ is a fuzzy vector subspace of } (V, \lambda)\}, \\ \Lambda &= \{L_{i,j}\} \cup \{L_{\langle r_i e_i + r_j e_j, e_k \rangle}\} \cup \{L_{\langle r_1 e_1 + r_2 e_2, r_1' e_1 + r_3 e_3 \rangle}\}, i \neq j, \end{aligned}$$

where

$$\begin{array}{rcl} L_{ij} : S & \longrightarrow & [0, 1] \\ < e_j > & \longmapsto & a_j \\ < e_i > & \longmapsto & a_i \\ U & \longmapsto & \min(a_i, a_j) \text{ for all } U \text{ of the form } \langle r e_i + s e_j \rangle \\ U & \longmapsto & 0 \text{ otherwise,} \end{array}$$

$$\begin{array}{rcl} L_{\langle r_i e_i + r_j e_j, e_k \rangle} : S & \longrightarrow & [0, 1] \\ \langle e_k \rangle & \longmapsto & a_k \\ \langle r_i e_i + r_j e_j \rangle & \longmapsto & \min\ (a_i, a_j) \\ U & \longmapsto & a_3 \text{ for all } U \text{ of the form } \langle a r_i e_i + b r_j e_j + c e_k \rangle \\ U & \longmapsto & 0 \text{ otherwise,} \end{array}$$

$$
\begin{array}{rcl}
L_{\langle r_1e_1+r_2e_2, r'e_1+r_3e_3\rangle}: S & \rightarrow & [0,1] \\
\langle r_1e_1+r_2e_2\rangle & \mapsto & a_2 \\
\langle r'_1e_1+r_3e_3\rangle & \mapsto & a_3 \\
U & \mapsto & a_3 \text{ for all } U \text{ of the form} \\
 & & \langle (ar_1+br'_1)e_1+ar_2e_2+br_3e_3\rangle \\
U & \mapsto & 0 \text{ otherwise,}
\end{array}
$$

for non-zero elements $r_i, a, b, c, r'_1 \in K$ and for $i, j, k = 1, 2, 3$.
The set π contains exactly the following types of fuzzy points:

$$\pi\{(\langle ae_1+be_2+ce_3\rangle, a_k)\},$$

where

$$a_k = \begin{cases} a_1 & \text{if} \quad b=c=0 \\ a_2 & \text{if} \quad c=0 \\ a_3 & \text{if} \quad c\neq 0. \end{cases}$$

First we show that π is complete. Let $U \in S$ then we have three cases:
Case 1. $U =< e_i >$, where $i = 1, 2, 3$. Hence $(\langle e_i\rangle, a_i) \in \pi$.
Case2. $U = \langle ae_i + be_j\rangle$, where $i \neq j = 1, 2, 3$ and $0 \neq a, b \in K$. In this case it is clear that $(U, a_2) \in \pi$ or $(U, a_3) \in \pi$.
Case 3. $U = \langle ae_i + be_j + ce_l\rangle$, where $i \neq j \neq l = 1, 2, 3$ and $0 \neq a, b, c \in K$. Hence $(U, a_3) \in \pi$.

Now we show that the $FPG(V)$ model satisfies the axioms F_1, F_2, F_3. First of all recall that any two finite dimensional vector spaces over K of the same dimension are isomorphic. Every fuzzy point in $FPG(V)$ is indeed a 1-dimensional subspaces of V, and two 1-dimensional subspaces determine a unique plane containing them, which is a line of $FPG(V)$. Then if $(\langle ae_1 + be_2 + ce_3\rangle, a_k)$ and $(\langle a'e_1 + b'e_2 + c'e_3\rangle, a'_k)$ are two fuzzy distinct points, the unique line containing them is the line $L_{\langle ae_1+be_2+ce_3, a'_1e_1+b'_2e_2+c'e_3\rangle}$. Note that if $a, b, c \neq 0$ and $a', b', c' \neq 0$, we can change the subspace $\langle ae_1+be_2+ce_3, a'e_1+b'e_2+c'e_3\rangle$ to a subspace of the form $\langle re_1+se_2, r'e_1+s'e_3\rangle$, where $r, s, r', s' \in K$, which depend on a, b, c, a', b', c'. Hence axiom F_1 holds.

Conversely, we show that for every two distinct fuzzy lines in Λ, there is a unique fuzzy point in π, with which both are incident. Since any two distinct 2-dimensional subspaces intersect each other in a unique 1-dimensional subspace, two distinct fuzzy lines in $FPG(V)$ intersect each other in a unique fuzzy point. Hence axiom F_2 holds. For details of how to find this intersection, see Appendix A.

It is clear that the four fuzzy distinct points $(\langle e_1\rangle, a_1)$, $(\langle e_2\rangle, a_2)$, $(\langle e_3\rangle, a_3)$, $(\langle e_1 + e_2 + e_3\rangle a_3)$ are such that no three of them are incident with one and the same line which validates axiom F_3. □

In the following theorem we assume that $a_1 > a_2 > a_3$.

Theorem 3.2. The fuzzy Desargues' proposition is valid in the $FPG(V)$ model.

Sketch of proof: Let $(\langle a_1e_1 + a_2e_2 + a_3e_3\rangle, a_k)$ be a fuzzy point. Consider all fuzzy lines passing through $(\langle a_1e_1 + a_2e_2 + a_3e_3\rangle, a_k)$ and choose three fuzzy lines from this collection such that we can construct Desargues' triangles with vertices on these lines with common point $(\langle a_1e_1 + a_2e_2 + a_3e_3\rangle, a_k)$. Then we can check that the theorem holds for all possible cases. For more details see Appendix B. □

Acknowledgement. The authors thank Ms. L. Kuijken for going through the manuscript and making valuable comments. Thanks are also due to the referee for his comments.

APPENDICES

Appendix A. It is clear that the lines $L_{i,j}$ and other fuzzy lines in Λ distinct from these lines $L_{i,j}$ are incident with a unique common point. For example we have $(\langle e_1\rangle, a_1)\ IL_{1,2}$ and $(\langle e_1\rangle, a_1)IL_{1,3}$. Therefore the two lines $L_{1,2}$ and $L_{1,3}$ are incident with the point $(\langle e_1\rangle, a_1)$.

Similarly the line $L_{\langle ae_i+be_j,e_k\rangle}$ and other fuzzy lines in Λ distinct from these lines, are incident with a con.mon point. For example:
$(\langle e_3\rangle, a_3)IL_{\langle ae_1+be_2,e_3\rangle}$ and $(\langle e_3\rangle, a_3)IL_{\langle a'e_1+b'e_2,e_3\rangle}$
$(\langle ae_1 + be_2 + \frac{bb'}{a}e_3\rangle, a_3)IL_{\langle ae_1+be_2,e_3\rangle}$ and $(\langle ae_1 + be_2 + \frac{bb'}{a}e_3\rangle, a_3)IL_{\langle a'e_2+b'e_3,e_1\rangle}$
$(\langle ae_1 + be_2 + \frac{ab'}{a'}e_3\rangle, a_3)IL_{\langle ae_1+be_2,e_3\rangle}$ and $(\langle ae_1 + be_2 + \frac{ab'}{a'}e_3\rangle, a_3)IL_{\langle a'e_1+b'e_3,e_2\rangle}$
$(\langle ae_1 + be_2 + \frac{d'(ab'-a'b)}{b'}e_3\rangle, a_3)IL_{\langle ae_1+be_2,e_3\rangle}$
and $(\langle ae_1 + be_2 + \frac{d'(ab'-a'b)}{b'}e_3\rangle, a_3)IL_{\langle a'e_1+b'e_2,c'e_1+d'e_3\rangle}$.

Finally we must show that the two distinct fuzzy lines $L = L_{\langle ae_1+be_2, ce_1+de_3\rangle}$ and $L' = L_{\langle a'e_1+b'e_2, c'e_1+d'e_3\rangle}$ are incident with a common point. To this end consider the following cases:

Case 1: $\frac{a}{a'} = \frac{b}{b'}$ and $\frac{c}{d} \neq \frac{c'}{d'}$.
In this case we have, $((\langle ae_1 + be_2\rangle, a_2)IL$ and $((\langle ae_1 + be_2\rangle, a_2)IL'$.

Case 2: $\frac{c}{d} = \frac{c'}{d'}$ and $\frac{a}{b} \neq \frac{a'}{b'}$.
In this case we have, $((\langle ce_1 + de_3\rangle, a_3)IL$ and $((\langle ce_1 + de_3\rangle, a)IL'$.

Case 3: $\frac{a}{b} \neq \frac{a'}{b'}$ and $\frac{c}{d} \neq \frac{c'}{d'}$.
In this case take $P = ((\langle\frac{bca'd'-adb'c'}{d'c-c'd}e_1 + bb'e_2 + \frac{(ba'-ab')dd'}{d'c-c'd}e_3\rangle, a_3)$. Then we have PIL and PIL', for if $r = 1, s = \frac{ba'-ab'}{d'c-c'd}$, then

$$\langle(ra + sc)e_1 + rbe_2 + sde_3\rangle = \langle\frac{bca'd' - adb'c}{d'c - c'd}e_1 + bb'e_2 + \frac{(ba' - ab')dd'}{d'c - c'd}e_3\rangle,$$

and if $r' = \frac{b}{b'}, s' = \frac{(ba'-ab')d}{d'c-c'd}$, then

$$\langle(r'a' + s'c')e_1 + r'b'e_2 + s'd'e_3\rangle = \langle\frac{bca'd' - adb'c'}{d'c - c'd}e_1 + bb'e_2 + \frac{(ba' - ab')dd'}{d'c - c'd}e_3\rangle.$$

Appendix B. Let $(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3, L_1L_2L_3)$ and $(\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3, L'_1L'_2L'_3)$ be two fuzzy triangles which satisfy the Desargues' condition. Firstly assume that the fuzzy lines connecting the corresponding vertices are incident with the fuzzy point $(\langle e_1\rangle, a_1)$. Then we have four cases:

Case 1: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$ be three lines $L_{1,2}$, $L_{1,3}$ and $L_{\langle ae_2+be_3, e_1\rangle}$.
In this case the possible triangles have their vertices in one of the following forms:
$\mathbf{x}_1 = ((\langle e_1\rangle, a_1)$ $\mathbf{x}_2 = ((\langle e_3\rangle, a_3)$ $\mathbf{x}_3 = ((\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_1\rangle, a_1)$ $\mathbf{x}_2 = ((\langle e_3\rangle, a_3)$ $\mathbf{x}_3 = ((\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_1\rangle, a_1)$ $\mathbf{x}_2 = ((\langle r'e_1 + s'e_3\rangle, a_3)$ $\mathbf{x}_3 = ((\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_1\rangle, a_1)$ $\mathbf{x}_2 = ((\langle r'e_1 + s'e_3\rangle, a_3)$ $\mathbf{x}_3 = ((\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = ((\langle e_1\rangle, a_1)$ $\mathbf{x}_3 = ((\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = ((\langle e_1\rangle, a_1)$ $\mathbf{x}_3 = ((\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = ((\langle e_1\rangle, a_1)$ $\mathbf{x}_3 = ((\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = ((\langle r'e_1 + s'e_3\rangle, a_3)$ $\mathbf{x}_3 = ((\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = ((\langle r'e_1 + s'e_3\rangle, a_3)$ $\mathbf{x}_3 = ((\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$
$\mathbf{x}_1 = ((\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = ((\langle e_3\rangle, a_3)$ $\mathbf{x}_3 = ((\langle e_1\rangle, a_1)$
$\mathbf{x}_1 = ((\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = ((\langle e_3\rangle, a_3)$ $\mathbf{x}_3 = ((\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$

$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle e_1\rangle, a_1)$ $\mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle e_1\rangle, a_1)$ $\mathbf{x}_3 = (\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle e_1\rangle, a_1)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle r'e_1 + s'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle e_1\rangle, a_1)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle r'e_1 + s'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle r'e_1 + s'e_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$

It is easy to show that if $(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3, L_1L_2L_3)$ and $(\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3, L'_1L'_2L'_3)$ are triangles of the above type and satisfy Desargues' hypothesis, then the theorem holds. For example if $\mathbf{x}_1 = (\langle e_1\rangle, a_1)$, $\mathbf{x}_2 = (\langle e_3\rangle, a_3)$, $\mathbf{x}_3 = (\langle ae_2+be_3\rangle, a_3)$ and $\mathbf{y}_1 = (\langle e_2\rangle, a_2)$, $\mathbf{y}_2 = (\langle e_1\rangle, a_1)$, $\mathbf{y}_3 = (\langle r''e_1 + s''ae_2 + s''be_3\rangle, a_3)$, then we have $L_1 = L_{2,3}, L'_1 = L_{1,3}, L_2 = L_{\langle ae_2+be_3,e_1\rangle}, L'_2 = L_{2,3}$ and $L_3 = L_{1,3}, L'_3 = L_{1,2}$, and we have $L_1 \cap L'_1 = \langle ae_2 + be_3\rangle, L_2 \cap L'_2 = \langle s''ae_2 + s''be_3 + r''e_1\rangle$ and $L_3 \cap L'_3 = \langle e_1\rangle$. Now it is clear that $\langle ae_2 + be_3\rangle$ and $\langle s''ae_2 + s''be_3 + r''e_1\rangle$ and $\langle e_1\rangle$ are colinear, because $\langle ae_2 + be_3\rangle$ and $\langle s''ae_2 + s''be_3 + r''e_1\rangle$ and $\langle e_1\rangle$ are incident with the line $L_{\langle ae_2+be_3,e_1\rangle}$.

Case2: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$ be the three lines

$$L_{1,2}, L_{\langle ae_2+be_3,e_1\rangle} \text{ and } L_{\langle a'e_2+b'e_3,e_1\rangle},$$

where $\frac{a}{a'} \neq \frac{b}{b'}$. In this case the possible triangles have their vertices in one of the following forms:

$\mathbf{x}_1 = (\langle e_1\rangle, a_1)$ $\mathbf{x}_2 = (\langle a'e_2 + b'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1)$ $\mathbf{x}_2 = (\langle a'e_2 + b'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle r''e_1 + sa''e_2 + sb''e_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1)$ $\mathbf{x}_2 = (\langle pe_1 + qa'e_2 + qb'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1)$ $\mathbf{x}_2 = (\langle pe_1 + qa'e_2 + qb'e_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle r''e_1 + sa''e_2 + sb''e_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle e_1\rangle, a_1)$ $\mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle e_1\rangle, a_1)$ $\mathbf{x}_3 = (\langle r''e_1 + sa''e_2 + sb''e_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle a'e_2 + b'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle e_1\rangle, a_1)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle a'e_2 + b'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle r''e_1 + sa''e_2 + sb''e_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle pe_1 + qa'e_2 + qb'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle e_1\rangle, a_1)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle pe_1 + qa'e_2 + qb'e_3\rangle, a_3)$ $\mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)$ $\mathbf{x}_2 = (\langle pe_1 + qa'e_2 + qb'e_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle r''e_1 + sa''e_2 + sb''e_3\rangle, a_3)$

$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2) \quad \mathbf{x}_2 = (\langle e_1\rangle, a_1) \quad \mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2) \quad \mathbf{x}_2 = (\langle e_1\rangle, a_1) \quad \mathbf{x}_3 = (\langle r''e_1 + sa''e_2 + sb''e_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2) \quad \mathbf{x}_2 = (\langle a'e_2 + b'e_3\rangle, a_3) \quad \mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2) \quad \mathbf{x}_2 = (\langle a'e_2 + b'e_3\rangle, a_3) \quad \mathbf{x}_3 = (\langle e_1\rangle, a_1)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2) \quad \mathbf{x}_2 = (\langle a'e_2 + b'e_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle r''e_1 + sa''e_2 + sb''e_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2) \quad \mathbf{x}_2 = (\langle pe_1 + qa'e_2 + qb'e_3\rangle, a_3) \quad \mathbf{x}_3 = (\langle e_1\rangle, a_1)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2) \quad \mathbf{x}_2 = (\langle pe_1 + qa'e_2 + qb'e_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle ae_2 + be_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle re_1 + se_2\rangle, a_2) \quad \mathbf{x}_2 = (\langle pe_1 + qa'e_2 + qb'e_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle r''e_1 + sa''e_2 + sb''e_3\rangle, a_3)$

It is easy to show that if $(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3, L_1L_2L_3)$ and $(\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3, L'_1L'_2L'_3)$ are triangles of the above types and satisfy Desargues' hypothesis, the theorem holds. The following cases can be treated similarly.

Case 3: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$ be the three lines

$$L_{1,3},\ L_{\langle ae_1+be_3,e_1\rangle} \text{ and } L_{\langle a'e_1+b'e_3,e_2\rangle},$$

where $\frac{a}{a'} \neq \frac{b}{b'}$.

Case 4: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$ be the three lines $L_{\langle ae_1+be_3,e_1\rangle}, L_{\langle a'e_1+b'e_3,e_1\rangle}$ and $L_{\langle a''e_1+b''e_3,e_1\rangle}$, where $\frac{a}{a'} \neq \frac{b}{b'}, \frac{a}{a''} \neq \frac{b}{b''}, \frac{a'}{a''} \neq \frac{b'}{b''}$.

One must note that the theorem is still valid even if the fuzzy lines connecting the corresponding vertices are incident with fuzzy point $(\langle e_2\rangle, a_2)$ or $(\langle e_3\rangle, a_3)$.

Secondly, assume that the fuzzy lines connecting the corresponding vertices are incident with the fuzzy point $(\langle ae_1 + be_2\rangle, a_2)$. Then we have three cases:

Case 1: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$ be the three lines

$$L_{1,2}, L_{\langle ae_1+be_2,e_3\rangle} \text{ and } L_{\langle ae_1+be_2,ce_1+de_3\rangle}.$$

In this case the possible triangles have their vertices in one of the following forms:

$\mathbf{x}_1 = (\langle e_1\rangle, a_1) \quad \mathbf{x}_2 = (\langle ae_1 + be_2\rangle, a_2) \quad \mathbf{x}_3 = (\langle ce_1 + de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1) \quad \mathbf{x}_2 = (\langle ae_1 + be_2\rangle, a_2)$

$\mathbf{x}_3 = (\langle (r'a + s'c)e_1 + r'be_2 + s'de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1)\quad \mathbf{x}_2 = (\langle e_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ae_1 + be_2\rangle, a_2)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1)\quad \mathbf{x}_2 = (\langle e_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle (r'a + s'c)e_1 + r'be_2 + s'de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1)\quad \mathbf{x}_2 = (\langle rae_1 + rbe_2 + se_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ae_1 + be_2\rangle, a_2)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1)\quad \mathbf{x}_2 = (\langle rae_1 + rbe_2 + se_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ce_1 + de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_1\rangle, a_1)\quad \mathbf{x}_2 = (\langle rae_1 + rbe_2 + se_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle (r'a + s'c)e_1 + r'be_2 + s'de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle ae_1 + be_2\rangle, a_2)\quad \mathbf{x}_3 = (\langle ce_1 + de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle ae_1 + be_2\rangle, a_2)$
$\mathbf{x}_3 = (\langle (r'a + s'c)e_1 + r'be_2 + s'de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle e_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ae_1 + be_2\rangle, a_2)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle e_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ce_1 + de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle e_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle (r'a + s'c)e_1 + r'be_2 + s'de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle rae_1 + rbe_2 + se_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ae_1 + be_2\rangle, a_2)$
$\mathbf{x}_1 = (\langle e_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle rae_1 + rbe_2 + se_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ce_1 + de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle pe_1 + qe_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle ae_1 + be_2\rangle, a_2)\quad \mathbf{x}_3 = (\langle ce_1 + de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle pe_1 + qe_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle ae_1 + be_2\rangle, a_2)$
$\mathbf{x}_3 = (\langle (r'a + s'c)e_1 + r'be_2 + s'de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle pe_1 + qe_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle e_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ae_1 + be_2\rangle, a_2)$
$\mathbf{x}_1 = (\langle pe_1 + qe_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle e_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ce_1 + de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle pe_1 + qe_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle e_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle (r'a + s'c)e_1 + r'be_2 + s'de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle pe_1 + qe_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle rae_1 + rbe_2 + se_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ae_1 + be_2\rangle, a_2)$
$\mathbf{x}_1 = (\langle pe_1 + qe_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle rae_1 + rbe_2 + se_3\rangle, a_3)\quad \mathbf{x}_3 = (\langle ce_1 + de_3\rangle, a_3)$
$\mathbf{x}_1 = (\langle pe_1 + qe_2\rangle, a_2)\quad \mathbf{x}_2 = (\langle rae_1 + rbe_2 + se_3\rangle, a_3)$
$\mathbf{x}_3 = (\langle (r'a + s'c)e_1 + r'be_2 + s'de_3\rangle, a_3)$

It is easy to show that if $(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3, L_1L_2L_3)$ and $(\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3, L_1', L_2'L_3')$ are triangles of the above types that satisfy Desargues' hypothesis, then the theorem holds.

The following cases can be treated similarly.

Case 2: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$, be the three lines $L_{\langle ae_1+be_2,e_3\rangle}$, $L_{\langle ae_1,be_2,ce_1+de_3\rangle}$ and $L_{\langle ae_1+be_2,c'e_1+d'e_3\rangle}$, where $\frac{c}{c'} \neq \frac{d}{d'}$.

Case 3: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$, be the three lines

$$L_{\langle ae_1+be_2, ce_1+de_3\rangle}, L_{\langle ae_1, be_2, c'e_1+d'e_3\rangle} \text{ and } L_{\langle ae_1+be_2, c''e_1+d''e_3\rangle},$$

where $\frac{c}{c'} \neq \frac{d}{d'}$, $\frac{c}{c''} \neq \frac{d}{d''}$, $\frac{c'}{c''} \neq \frac{d'}{d''}$.

One notes that the theorem is still valid even if the fuzzy lines connecting the corresponding vertices are incident with the fuzzy point $(\langle ae_2 + be_3\rangle, a_3)$ or $(\langle ae_1 + be_3\rangle, a_3)$.

Thirdly, assume that the fuzzy lines connecting the corresponding vertices are incident with the fuzzy point $(\langle ae_1 + be_2 + ce_3\rangle, a_3)$, then we have the following two cases which can be treated similarly:

Case 1: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$, be three the lines $L_{\langle ae_1+be_2, e_3\rangle}$, $L_{\langle ae_1+be_2, ce_1+de_3\rangle}$ and $L_{\langle be_2+ce_3, e_1\rangle}$.

Case 2: Let $\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_3$, be three the distinct lines $L_{\langle a'e_1+b'e_2, c'e_1+d'e_3\rangle}$, $L_{\langle ae_1+be_2, ce_1+de_3\rangle}$ and $L_{\langle a''e_1+b''e_2, c''e_1+d''e_3\rangle}$.

REFERENCES

1. A. Bogomolny, *On the perimeter and area of fuzzy sets*, Fuzzy Sets and Systems (1987), 257-269.

2. J.J. Buckley and E. Eslami, *Fuzzy plane geometry I, points and lines*, Fuzzy Sets and Systems, **86** (1997), 179-187

3. J.J. Buckley and E. Eslami, *Fuzzy plane geometry II, circles and polygons*, Fuzzy Sets and Systems, **87** (1997), 79-85.

4. S. C. Chang and J. N. Mordeson and Yu.Yandong, *Elements of L-Algebra*, Lecture notes in fuzzy mathematics and computer science, Center for research in fuzzy mathematics and computer science, Creighton University, Omaha, Nebraska, USA, 1994.

5. K.C. Gupta and S. Ray, *Fuzzy plane projective geometry*, Fuzzy Sets and Systems **54**(1993), 191-206.

6. D.R. Hughes and F.C.Piper, *Projective Planes,* Springer-Verlag , Berlin, 1973.

7. L. Kuijken, H. Van Maldeghem and E. Kerre, *Fuzzy projective geometry from fuzzy vector spaces*, Proceedings of IPMU, Paris, Vol.II(1998), 1331-1338.

8. L. Kuijken, H. Van Maldeghem and E. Kerre, *Fuzzy projective geometry from fuzzy groups*, Tatra Mountains Mathematical Publications, to appear.

9. J. N. Mordeson, *L-Subspaces and L-subfields*, Lecture notes in fuzzy mathematics and computer science, Center for research in fuzzy mathematics and computer science, Creighton University, Omaha, Nebraska, USA, 1996.

10. J. N. Mordeson and P. S. Nair, Fuzzy mathematics, *An Introduction For Engineers and Scientist*, Studies in Fuzziness and Soft Computing, Vol. 20, Physica-Verlag, Springer-Verlag , Berlin, 1998.

11. F. Nemat and E. Eslami, *Fuzzy space geometry I, points, lines and planes*, Journal of Fuzzy Mathematics, Vol.9, **3** (2001) 659-675.

12. F. Nemat and E. Eslami, *Fuzzy space geometry II, subpoints and sublines*, Journal of Fuzzy Mathematics, Vol.9, **3** (2001), 693-700.

13. F. Nemat, *Fuzzy Space Geometry*, Ph.D. Dissertation, Kerman University (2002).

14. A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512-517.

15. A. Rosenfeld, *Some results on fuzzy (digital) convexity*, Pattern Recognition, Vol. 15, **5** (1992), 379-382.

16. A. Rosenfeld, *The diameter of a fuzzy set*, Fuzzy Sets and Systems **13** (1984), 241-382.

17. A. Rosenfeld, *The perimeter of a fuzzy set*, Pattern Recognition **18** (1985), 125-130

18. A. Rosenfeld, *Distances between fuzzy sets*, Pattern Recognition Letters **3**(1985), 229-233.

19. A. Rosenfeld, *Fuzzy rectangles*, Pattern Recognition Letters **11**(1990), 677-679.

20. A. Rosenfeld, *Fuzzy plane geometry*, Triangles, Proceedings of Third Int. Conf. on fuzzy system, Orlando, June 26-69, Vol. 2 (1994), 891-893.

21. L. A. Zadeh, *Fuzzy Sets*, Inform. Control **8** (1965), 338-353.

Islamic Azad University of Kerman, Kerman, Iran.

Faculty of Mathematics and Computer Sciences, Kerman University, Kerman, Iran

Faculty of Mathematics and Computer Sciences, Kerman University, Kerman, Iran
e-mail: mashinchi@mail.uk.ac.ir

Date received January 23, 2002