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THE LIE ALGEBRAIC TREATMENT OF PARTIAL DIFFERENTIAL EQUATIONS OF EVOLUTION TYPE

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Abstract. In this paper we discuss in some detail how the Lie algebra method may be utilized to solve partial differential equations of evolution type. Of particular interest is the Schrödinger equation. Several examples of this type, illustrating the method, have been introduced and their solutions are derived. More precisely we have considered SU(1, 1) and SU(2) Lie algebra quantum systems. Furthermore we have also discussed the possibility of employing quadratic invariants to obtain different classes of the wave functions.

1. Introduction

Lie algebras and their associated lie groups are closely related to differential equations and their corresponding solutions. Historically speaking, the deep reasons underlying the integrability by quadratures of ordinary differential equations was first explored by Lie via his group. The second order ordinary differential equation give birth to the theory of functions, while the special functions were obtained [1] as matrix elements of operators defining irreducible group representations. In the meantime Lie algebraic methods have been used for computing eigenvalues and recurrence relation [2]. The algebraic treatment of partial differential equations is deeply rooted. For instance Schrödinger applied the method of factorization to solve the time-independent Schrödinger equation. Later, Miller showed that this method is equivalent to the representation theory of four Lie algebras [3]. In fact Miller presented

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further examples in a systematic manner in his book [4]. S.Steinberg had also utilized Lie algebraic methods to obtain explicit solutions of the partial differential equation, see for example [5]. He considered initial value problems for differential operators L:

$$\frac{\partial}{\partial t} f(x,t) = Lf(x,t), \qquad f(x,0) = g(x) \tag{1.1}$$

whose solution can be written in the form

$$f(x,t) = g(x)\exp(Lt) \tag{1.2}$$

Recently G. Dattoli et al.[6, 7] took a further step in the direction of obtaining explicit solutions of the partial differential equation of evolution type similar to equation (1.1). They exploit matrix realizations of the basis elements for a closed Lie algebra G whose associated Lie group is then obtained in a matrix form.

In fact Steinberg and Dattoli et al idea underlying solving partial differential equations (P.D.E.) via Lie algebra can be synthesized as follows: Given a P.D.E. of type (1.1) one may identify a basis for a closed Lie algebra of operators to which L belongs. If L_i is this basis, then one can use ordering formulas of Baker,Haussdorff, Campbell and Zassenhaus to write the operator $\exp(Lt)$ appearing in equation (1.2) in the form

$$\exp(Lt) = \prod_{i=1}^{n} \exp(L_i S_i(t)) \tag{1.3}$$

where $S_i(t)$ are time-dependent functions. Once the functions $S_i(t)$ are determined, then the explicit solution of equation (1.1) can be found using simple operational rules involving successive actions of $\exp(L_i S_i(t))$, $1 \leq i \leq n$, on the initial condition $g(x)$. In particular, following Dattoli et al, one may use a matrix representation $\acute{G}$ of the evolution group G, in order to determine the functions $S_i(t)$. This can be done by writing L as a linear combination of the basis elements of the assigned Lie algebra and using equation (1.1) to obtain a matrix image of this equation. The Lie matrix group $\acute{G}$, which is in fact isomorphic to the evolution group G, is then found from the matrix image of the P.D.E. i.e. the evolution operator $\exp(Lt)$ is determined in a matrix form. Then equation (1.3) is used to calculate the functions $S_i(t)$. It should be noted that the closed Lie algebra basis will be chosen such that $L = \sum_{i=1}^{n} \alpha_i L_i$.

Therefore, if one uses equation (1.3) and manages to determine the functions $S_i(t)$, then the evolution group will be known. This means that the solution of the differential equation can be obtained by successive action of the elements.

In this paper we discuss in some detail the theory behind the method where we emphasize the close relationship of Lie algebras and their associated Lie groups to differential equations and their corresponding solutions. A historical problem is to find the 1-dimensional subgroup corresponding to the 1-dimensional algebra (i.e. Hamiltonian) of the vector field arising from the differential equation. Although the existence and uniqueness of the solution of this problem is guaranteed up to group isomorphism, there is no general systematic technique to obtain solutions of P.D.E's from the classical point of view. In the meantime the vector field associated to the differential equation is a 1-dimensional subalgebra of the (infinite dimensional) Lie algebra of all vector fields on the phase manifold. Therefore, we regard a differential equation on the manifold of phase space as a vector field and the solution of this equation as a 1- parameter group acting on the initial point on the manifold. This is seen in section **II** in addition we intend to give some account of the various ways one may calculate the functions $S_i(t)$ in the product given by equation (1.3) which are the key points of this method. In section **III** we call attention to a direction towards a vision of obtaining solutions of P.D.E's by a systematic method. To emphasize this point, roughly speaking, the method described in section **II** is to rediscover the evolution group $\exp(Lt)$ of the differential equation (1.1) in terms of a product of simple group elements $\exp(L_i S_i(t))$ corresponding to a closed Lie algebra basis (L_i). In section **IV** we introduce several examples where this method can be successfully utilized. Some of these examples have been previously treated [8, 9, 10, 11, 12, 13], however, it will be more convenient to outline them here in order to bring to light the power of the Lie algebraic method and it's richness. We give our conclusion in section V.

2. Differential equations on Manifold

Since the equations of motion of a physical system are differential equations, the solutions will be curves having a preferred parametrization. In situations where the preferred parametrization are not desired, one replaces the differential equation (which is in fact a vector field) with a one-dimensional foliation.

2.1. **Differential equations as 1-dimensional foliations on manifold.** A one-dimensional foliation is a pair (M, E) where M is a manifold and E is a function which assigns to each $x \in M$ a one-dimensional subspace E_x of the tangent space $T_x M$ at x. E should be smooth in the sense that there locally be a smooth non-zero vector field X with $X(x)$ generating E_x. The leaf (an orbit) of the foliation (M, E) is a connected, one-dimensional submanifold φ of M such that $T_x\varphi = E_x$ for $x \in M$ and maximal with respect to these properties. Through each point there exists a unique orbit, (existence and uniqueness theorem for differential equations) but these orbits have no preferred parameterization. A time-dependent vector field on a manifold M is a map $X : \mathcal{R} \times M \to TM$ such that $X(t, x) \in T_x M$ where $T_x M$ is the tangent space bundle of M and $x \in M$. The integral curves of the vector field X are the solutions $\varphi(t)$ of the differential equation

$$\frac{d}{dt}\varphi(t) = X(t, \varphi(t)) \tag{2.1}$$

which have a preferred parametrization. The foliation $(\bar{M}, E)$ given by

$$\bar{M} = \mathcal{R} \times M \qquad \textit{and} \qquad E_{(t,x)} = \{x, X(t, x) : x \in \mathcal{R}\}, \qquad (t, x) \in \bar{M} \tag{2.2}$$

is called the suspension of X; its orbits are parametrized by the curve $t \to (t, \varphi(t))$ where φ is an integral curve of X. Now suppose $\bar{M}/_E$ denote the set of orbits of the one-dimensional foliation $(\bar{M}, E)$. In this case it sometimes happen that $\bar{M}/_E$ admits the structure of a smooth manifold in such a way that the natural projection $\bar{M} \to \bar{M}/_E$ is a smooth submersion (surjective derivative at each point of $\bar{M}$).

2.2. **Lie groups and Lie algebras.** A Lie group G is a smooth manifold which is also a group such that the group operations are smooth. A Lie algebra is a real vector space $\mathcal{G}$ together with a bilinear skew-symmetric map: $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : (A, B) \to [A, B]$ which satisfies Jacobi's identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{2.3}$$

where $A, B, C \in \mathcal{G}$. Let us now introduce the following lemma which is needed for subsequent discussion.

Lemma: Let G be a Lie group and $A \in T_eG$ the tangent space to G at the identity e . In this case there is a unique homomorphism of Lie groups: $\mathcal{R} \to G : t \to \exp(tA)$,such that $\frac{d}{dt}\exp(tA)|_{t=0} = A$.(Note that the homomorphism of Lie group should be smooth). The Lie algebra of G has as underlying vector space the tangent space T_eG at e, and brackets operation given by

$$(2.4) \qquad [A, B] = \frac{d}{dt}\exp(tA)\exp(tB)\exp(-tA)\exp(-tB)|_{t=0}$$

where $A, B \in T_eG$, which defines a Lie algebra. When G is the group of all linear automorphism of $\mathcal{R}^n$, then $\mathcal{G}$ is the algebra of all linear endomorphisms of $\mathcal{R}^n$ with brackets operations: $[A, B] = AB - BA$. Now let G and $\acute{G}$ be two Lie groups which are homomorphic. If $f : G \to \acute{G}$ is the homomorphism, then one can define the associated Lie algebra homomorphism $F : \mathcal{G} \to \acute{\mathcal{G}}$ (i.e.$F[A, B] = [F(A), F(B)]$) by

$$(2.5) \qquad F(A) = \frac{d}{dt}f(\exp(tA))|_{t=0} \qquad \text{for any } A \in G$$

so that $\exp(tF(A)) = f(\exp(tA))$, from which one can easily see that F is a Lie algebra homomorphism. Here we may point out to an important relation between Lie algebras and Lie groups. This relation appeared from the existence and uniqueness theorem, that is:

(i) Every finite dimensional Lie algebra G is the Lie algebra of a simply connected Lie group G; moreover it is unique to Lie group isomorphism.

(ii) Every homomorphism $F : \mathcal{G} \to \acute{\mathcal{G}}$ of Lie algebras is the associated Lie algebra homomorphism of a unique homomorphism $f : G \to \acute{G}$ of Lie groups. Here we may point out that one proves (i) from (ii). Existence in (i) follows from Ado's theorem [14] and the correspondence between Lie subalgebras and Lie subgroups [15].

2.3. **The group of self-diffeomorphisms of a manifold.** One can think of the group $Diff(M)$ of self-diffeomorphisms of a manifold M as an " infinite dimensional Lie group". To see this let $t \to f_t \in Diff(M)$ be a curve of diffeomorphisms with f_0 =identity. Then for $x \in M$ we have

$$(2.6) \qquad \frac{d}{dt}f_t(x)|_{t=0} = X(x)$$

i.e. $X(x) \in T_x M$ and therefore X is a vector field on M. Hence the Lie algebra of $Diff(M)$ is the Lie algebra of vector fields on M. To get the Lie group corresponding to this Lie algebra, consider the group homomorphism $\mathcal{R} \to Diff(M) : t \to f^t$ generated by $X \in X(M)$. This group homomorphism is the flow of X; i.e. the unique solution of the differential equation

$$\frac{d}{dt} f^t(x) = X\left(f^t(x)\right), \qquad f^0(x) = x. \tag{2.7}$$

Now we go further on the analogy between Lie groups and diffeomorphism groups. Let G be a Lie group and M a manifold. An action of G on M is a homomorphism of groups: $G \to Diff(M), a \to a_M$ such that the evaluation map: $G \times M \to M : (a, x) \to a_M(x)$ is smooth. A G-manifold is a pair consisting of a manifold M and an action of G on M. An action of the Lie algebra $\mathcal{G}$ on M is a homomorphism of Lie algebras: $\mathcal{G} \to \mathcal{G}(M)$: $A \to A_M$ such that the evaluation map: $\mathcal{G} \times M \to M : (A, x) = A_M(x)$ is smooth. A $\mathcal{G}$-manifold is a pair consisting of a manifold together with an action of $\mathcal{G}$ on M. Given a $\mathcal{G}$-action on M one may form the associated $\mathcal{G}$-action via

$$A_M(x) = \frac{d}{dt} \exp{(tA)}_M (x)|_{t=0}, \qquad x \in M, \qquad A \in \mathcal{G} \tag{2.8}$$

Then we have the following:

Proposition: Let G be a simply connected Lie group and M a compact manifold. Then every action of $\mathcal{G}$ on M is associated with a unique action of G on M, see Ref.[16].

It should be noted that the compactness hypothesis is essential because vector fields on non-compact manifolds need not be complete and therefore its flow need not be defined for all t and x. Now we can consider the group of linear automorphisms of the vector space $\mathcal{H}(M)$ of smooth real-valued functions on M. There is a natural antihomomorphisms from $Diff(M)$ to this group i.e. $f \to f^*$ such that $f^* h = h.f$ for $h \in \mathcal{H}(M)$. The associated antihomomorphisms of Lie algebras, denoted by l, sends $\mathcal{X}(M)$ to the space of linear endomorphism of $\mathcal{H}(M)$. Thus

$$\left(l\left(X\right)h\right)(x) = \frac{d}{dt} f^{*t} h(x)|_{t=0} \tag{2.9}$$

for $x \in M, h \in \mathcal{H}, X \in \mathcal{X}(M)$ and where f is the flow of X. In this case $l(X)$ is Lie differentiation. If A and B are endomorphisms of a vector space such that their Lie brackets is $[A, B] = AB - BA$, then l is an antihomomorphism:

$$l([X, Y]) = -[l(X), l(Y)], \qquad X, Y \in \mathcal{X}(M) \tag{2.10}$$

In the next section we turn our attention to the Lie algebraic treatment for partial differential equations.

3. Lie algebraic treatment

We devote this section to the partial differential equation of evolution type from the Lie algebraic point of view. In the meantime we shall introduce the analysis of Lie algebra itself.

3.1. **Partial differential equations of evolution type.** A prototype of the above mentioned equations is the following Cauchy problem:

$$\begin{aligned} \frac{\partial}{\partial t} f(t, x_1, x_2, ...x_n) &= L(x_1, x_2, ...x_n) f(t, x_1, x_2, ...x_n), \\ f(t, x_1, x_2, ...x_n)|_{t=0} &= g(x_1, x_2, ...x_n), \end{aligned} \tag{3.1}$$

where its solution can be written in the form

$$f(x, t) = \exp(Lt)\, g(x), \qquad x = (x_1, x_2, ...x_n) \tag{3.2}$$

It should be noted that the operator L in general depends on t, but for simplicity we ignore the time-dependence bearing in mind that our discussion can be extended without any problem to the case in which L depends on t. Since our differential equation is of evolution type, we assume that the operator L is an element of a Lie algebra $\mathcal{G}$ of some symmetry group G of the dynamical system. Let L_i be a basis for $\mathcal{G}$. Then

$$L = \sum_{k=1}^{m} a_k L_k \tag{3.3}$$

provided that the Lie algebra is closed and we have

$$[L_i, L_j] = \sum_{k=1}^{n} C_{ij}^k L_k, \tag{3.4}$$

This condition also implies that G is isomorphic to some matrix Lie algebra $\acute{\mathcal{G}}$. Since $L \in \mathcal{G}$ which is a closed Lie algebra, one may use the Lemma in the previous section to interpret $\exp(Lt)$ in equation (3.2) as an element in the Lie group associated with $\mathcal{G}$, i.e.the equation (3.2) can be explicitly written as

$$f(x_i, t) = \exp(Lt)\, g(x_i) = l(t) g(\varphi_i(t)) \tag{3.5}$$

where $l(t)$ and $\varphi_i(t)$ are time-dependent functions that can be determined. To see that we write the basis L_i for the algebra $\mathcal{G}$ as

$$L_i = \sum_{j=1}^{n} k_{i,j}(x_1, x_2, ...x_n) \frac{\partial}{\partial x_j} + r_i(x_1, x_2, ...x_n) \tag{3.6}$$

from which we can easily prove that both $l(t)$ and $\varphi_i(t)$ are solutions of

$$\begin{aligned} \frac{d}{dt} l(t) &= l(t) \sum_{i=1}^{n} a_i r_i(\varphi(t)) \qquad l(0) = 1 \\ \frac{d}{dt} \varphi_i(t) &= \sum_{i=1}^{n} a_i k_{i,j}(\varphi_1(t), ..., \varphi_n(t)), \qquad \varphi_j(0) = x_j \end{aligned} \tag{3.7}$$

Thus the operator $L \in \mathcal{G}$ is a generator of a one-parameter group $l(t)$, $t \in \mathcal{R}$. This is in fact the case in which the operator L is related to the Hamiltonian operator $H = -iL$. In this particular case we have the Schrödinger equation describing the evaluation of the wave function, i.e. $\partial\psi(t)/\partial t = -iH\psi(t)$. The evoluation group $l(t)$ will be similarly given by $l(t) = \exp(-iHt)$. The Schrödinger equation will be of particular interest for us in the forthcoming section. However, let us complete this section by considering the systematic Lie algebra method of obtaning the solution for the partial differential equations of type (3.1).

3.2. **Analysis of Lie algebra method.** As we have seen the solution of equation (3.1) is determined by equation (3.2),i.e.via $\exp(Lt)$. Suppose now that a closed Lie algebra basis is determined such that the operator L is written as in equation (3.3). The Lie group corresponding to this Lie algebra is a one-parameter group determined by the evolution operator $\exp(Lt)$. Then we can write

$$\exp(Lt) = \prod_{i=1}^{m} \exp(S_i(t) L_i), \tag{3.8}$$

where $S_i(t)$ are functions ot t linked to the coefficients of L. The above splitting is possible due to the ordering formulas of Baker, Hausdorff,Campbell and Zassenhaus, see for example ref.[7, 17, 18]. To determine the functions $S_i(t)$ we have to use a matrix realization for the Lie algebra basis L_i to write equation (3.1) in matrix form. From this matrix image of the equation (3.1) we obtain a matrix realization $\acute{G}$ of the evoluation group G. However, this matrix realization is isomorphic to G and consequentely this matrix group being $\exp{(Lt)}$ is equated to the right hand side of equation (3.8) in order to find the functions $S_i(t)$. This is provided that the product $\prod_{i=1}^{m} \exp{(S_i(t)L_i)}$ is expressed in a matrix closed form. Once the functions $S_i(t)$ are thus found, we have to go back to the generators written in G and find the right hand side of equation (3.8). Hence the action of the evoluation operator $\exp{(Lt)}$ on the initial condition g(x) is determined and thus the explicit solution of the partial differetial equation is found.

Now we turn our attention to discuss the key point of the Lie algebra method, namely the search for an ordered product of the type (3.8). There are two ways to write the basis elements L_i of the Lie algebra. One as a differential operator and the other as a matrix form, which will be discuss in the following subsection.

3.2.1. *The matrix form treatment.* If the basis elements L_i of the Lie algebra are written in a matrix form, then a typical element of the product in equation (3.8) say $\exp(a)$ is to be calculated in a closed matrix form ($a = S_i(t)L_i$ for some i). In fact $\exp(a) = \sum_{n=0}^{\infty} a^n/n!$ which is not easily manageable, so we need to have $\exp(a)$ in a closed matrix form. To accomplish this we suppose that the operator function $\exp(a)$ acts on an n-dimensional vector space. If $\psi(t)$ is an n-column vector, then one can write

$$\frac{d}{dt}\psi(t) = a\psi(t), \qquad \text{i.e.} \qquad \frac{d}{dt}\psi_i(t) = \sum_{j=1}^{n} a_{ij}\psi_j(t) \tag{3.9}$$

The above equation is of evolution type and its solution can be written formaly as $\psi(t) = A(t)\psi(0)$ where $A(t)$ is an $n \times n$ matrix and $\psi(0)$ are initial conditions. Therefore after a direct substitutaion of $\psi(t) = A(t)\psi(0)$ into equation (3.9), thus we have

$$\frac{d}{dt}A_{ij} = \sum_{j=1}^{n} a_{ik}A_{kj}, \qquad A_{ij}(0) = \delta_{ij}, \tag{3.10}$$

which represents a system of linear differential equations that specifies the matrix $A(t)$. It should be noted that for $\exp(a) = A(1)$ one is able to obtain $\exp(a)$ in a closed form. Also we may point out that for 2×2 matrices the solution of eqution (3.10) can be written as

$$\begin{aligned} A_{11} &= \left(\frac{1}{\sqrt{\Delta}}(a_{11} - a_{22})\sinh\left(\frac{\sqrt{\Delta}}{2}\right) + \cosh\left(\frac{\sqrt{\Delta}}{2}\right)\right)\exp\left(\frac{1}{2}Tra\right) \\ \frac{A_{12}}{a_{12}} &= \frac{A_{21}}{a_{21}} = \frac{2}{\sqrt{\Delta}}\sinh\left(\frac{\sqrt{\Delta}}{2}\right)\exp\left(\frac{1}{2}Tr(a)\right) \\ A_{22} &= \left(-\frac{1}{\sqrt{\Delta}}(a_{11} - a_{22})\sinh\left(\frac{\sqrt{\Delta}}{2}\right) + \cosh\left(\frac{\sqrt{\Delta}}{2}\right)\right)\exp\left(\frac{1}{2}Tra\right) \end{aligned} \tag{3.11}$$

where $Tra = a_{11} + a_{22}$ and $\Delta = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$. To obtain a general method for finding the coefficients $S_i(t)$ in equation (3.8) we proceed as follows: Let A be a matrix representation for the operator L and A_i be matrix representations for the basis operators L_i for each i. Then we have

$$\exp\left(\hat{A}t\right) = \prod_{i=1}^{n} \exp\left(S_i(t)\hat{A}_i\right). \tag{3.12}$$

Differentiating with respect to t and after minor algebra we get

$$\hat{A} = \sum_{j=1}^{n}\left[\prod_{i=1}^{j-1}\exp\left(S_i(t)\hat{A}_i\right)\acute{S}_j(t)\hat{A}_j\prod_{i=j}^{n}\exp\left(S_i(t)\hat{A}_i\right)\right]\exp\left(-\hat{A}t\right). \tag{3.13}$$

To simplify the above equation we need to find a matrix C such that

$$\exp\left(S_i(t)\hat{A}_i\right)\hat{A}_j\exp\left(-S_i(t)\hat{A}_i\right) = C. \tag{3.14}$$

In fact it is well known that for any operator $\hat{A}$ in the Lie algebra $\mathcal{G}$, the operator $B \to \exp(A)B\exp(-A)$ is linear and thus has a matrix represenatation with respect to the matrix basis $\{A_i\}$ for $\mathcal{G}$. Therefore, for any $\alpha \in C$

and $C^{(i)}(\alpha) = C^{(i)}_{kj}(\alpha)$ we have

$$\exp\left(S_i(t)\hat{A}_i\right)\hat{A}_j\exp\left(-S_i(t)\hat{A}_i\right) = \sum_k C^i_{kj}\hat{A}_k = C^i_j \tag{3.15}$$

where

$$C^i_j = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\hat{A}_i,\hat{A}_j\right]^n, \qquad \left[\hat{A}_i,\hat{A}_j\right]^n = \left[\hat{A}_i,\left[\hat{A}_i,\hat{A}_j\right]^{n-1}\right]. \tag{3.16}$$

Furthermore, for any operator $\hat{A}$ in the Lie algebra $\mathcal{G}$, ad_A defined by $ad_A B = [A,B]$ is also a linear operator on $\mathcal{G}$ and consequently can be presented by a matrix $M_A \sim ad_A$ with respect to the matrix basis $\{A_i\}$ for $\mathcal{G}$, from which we have the special matrices $M^{(i)} = M^{(i)}_{jk}$ defined by $ad_A\left(A_k\right) = \sum_j M^{(i)}_{jk}A_j$. In this case equation (3.16) gives $C^{(i)}(\alpha) = \exp\left(\alpha M^{(i)}\right)$ and hence if we reduce $M^{(i)}$ to Jorden form we see that the elements of $C^{(i)}$ are polynomials in α. Now let us define the following elements of G

$$\begin{aligned} X_{0,k} &= A_k, \qquad \tilde{X}_{n+1,k} = A_k, \\ X_{j,k} &= \left(\prod_{i=1}^{j}\exp\left(x_iA_i\right)\right)A_k\left(\prod_{i=j}^{n}\exp\left(-x_iA_i\right)\right), \\ \tilde{X}_{j,k} &= \left(\prod_{i=1}^{j}\exp\left(-x_iA_i\right)\right)A_k\left(\prod_{i=j}^{n}\exp\left(x_iA_i\right)\right), \end{aligned} \tag{3.17}$$

where $X = (x_1, x_2, ..., x_n) \in C^n$, $0 \le j \le n+1, 1 \le k \le n$. Using equation (3.15) we obtain

$$X_{j,k} = \sum_l C^j_{l,k}(x_j)X_{j-1,l} \qquad \tilde{X}_{j,k} = \sum_l C^j_{l,k}(-x_j)X_{j+1,l} \tag{3.18}$$

Now if we assume that

$$\begin{aligned} X_{j,k} &= \sum_l B^j_{l,k}(x)A_l, \qquad B^j(x) = \left(B^j_{l,k}(x)\right), \\ \tilde{X}_{j,k} &= \sum_l \tilde{B}^j_{l,k}(x)A_l, \qquad \tilde{B}^j(x) = \left(\tilde{B}^j_{l,k}(x)\right), \end{aligned} \tag{3.19}$$

thus we have $B^{(0)} = \tilde{B}^{(n+1)} = I$ and

$$\begin{aligned} B^{(j)}(x) &= C^{(1)}(x_1)C^{(2)}(x_2)....C^{(j)}(x_j) \\ \tilde{B}^{(j)}(x) &= C^{(n)}(-x_n)C^{(n-1)}(-x_{n-1})....C^{(j)}(-x_j), \end{aligned} \tag{3.20}$$

Now we can use equation (3.19) to write equation (3.15) in the form

$$A = \sum_{j=1}^{n} \acute{S}_j X_{j-1,j}(S) = \sum_{j=1}^{n} \acute{S}_j \sum_{l} B_{l,j,j}^{(j-1)}(S) A_l, \qquad 1 \leq l \leq n \tag{3.21}$$

i.e. if $A = \sum a_l A_l$, then

$$a_l = \sum_{j=1}^{n} B_{l,j}^{(j-1)}(S) \acute{S}_j \tag{3.22}$$

gives us a set of ordinary differential equations describing the required set of functions $\{S_i\}$.

Before we close this section it is interesting to refere to another way to deal with such problem, that is the differential operator method. The method essentially depends on equations (3.3), (3.5),(3.6) and (3.7) and to see that let us consider the action of the exponential operator $\exp(Lt)$ on the function $g(x)$ (say). This gives us

$$\exp\left(a\frac{d}{dx}\right) g(x) \exp\left(-a\frac{d}{dx}\right) = g(x+a), \tag{3.23}$$

which is trival. Also we can prove that

$$\exp\left(ax\frac{d}{dx}\right) g(x+b) \exp\left(-ax\frac{d}{dx}\right) = g(e^a x + b). \tag{3.24}$$

To illustrate the previous method we turn our attention to some quantum system and find the most general wave function for them. This will be seen in the following section.

4. The wave function for some quantum system

We devote the present section to introduce some quantum system as an applications to the theory exhibited in the previous sections. For this reason we shall consider five examples based on the Hamiltonian of quantum systems. This will be in addition to the possibility for using the quadratic invariant for the Hamiltonian itself, which in fact opens the way to produce other classes of the wave functions. However, this will be in a separate section

4.1. **The frequency converter model.** The first example we introduce in this section is the frequency converter model [19]. The model is $\mathbf{SU}(2)$ Lie algebra and can be obtained either from the variation in the susceptibility during the interaction between the cavity modes or from the transformation of Tavis-Cummings model which represents the interaction between the atom and field within cavity [20]. The Hamiltonian which describes such a model is given by

$$\frac{\hat{H}}{\hbar} = \omega_1 \hat{a}^\dagger \hat{a} + \omega_2 \hat{b}^\dagger \hat{b} + \lambda(t)\left(\hat{a}\hat{b}^\dagger \exp\left(i\phi(t)\right) + \hat{b}\hat{a}^\dagger \exp\left(-i\phi(t)\right)\right), \tag{4.1}$$

where $\hat{a}^\dagger(\hat{a})$ and $\hat{b}^\dagger(\hat{b})$ are boson creation and annihilition operators for the first and second mode, respectively. These operators satisfy the commutation relations

$$\left[\hat{a}, \hat{a}^\dagger\right] = \left[\hat{b}, \hat{b}^\dagger\right] = 1, \qquad \left[\hat{a}, \hat{b}\right] = \left[\hat{a}, \hat{b}^\dagger\right] = \left[\hat{a}^\dagger, \hat{b}^\dagger\right] = 0. \tag{4.2}$$

Also the frequencies $\omega_i, i = 1, 2$ are the field frequency and $\lambda(t)$ is the time-dependent coupling parameter while $\phi(t)$ is the time-dependent phase pump. By introducing the generators

$$\hat{K}_1 = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}, \qquad \hat{K}_2 = \hat{b}^\dagger \hat{b}, \qquad \hat{K}_3 = \hat{a}\hat{b}^\dagger, \qquad \hat{K}_4 = \hat{a}^\dagger \hat{b} \tag{4.3}$$

which satisfy the commutation relations

$$\begin{aligned} \left[\hat{K}_1, \hat{K}_2\right] &= 0, \qquad \left[\hat{K}_1, \hat{K}_3\right] = -2\hat{K}_3, \qquad \left[\hat{K}_1, \hat{K}_4\right] = 2\hat{K}_4, \\ \left[\hat{K}_2, \hat{K}_3\right] &= \hat{K}_3, \qquad \left[\hat{K}_2, \hat{K}_4\right] = -\hat{K}_4, \qquad \left[\hat{K}_4, \hat{K}_3\right] = \hat{K}_1 \end{aligned} \tag{4.4}$$

In this case we can rewrite the Hamiltonian thus

$$\frac{\hat{H}}{\hbar} = \omega_1 \hat{K}_1 + (\omega_1 + \omega_2)\hat{K}_2 + \lambda(t)\left(\hat{K}_3 \exp\left(i\phi(t)\right) + \hat{K}_4 \exp\left(-i\phi(t)\right)\right), \tag{4.5}$$

and therefore if we solve the Schrödinger equation

$$\hat{H}(t)\psi(t) = i\hbar\frac{\partial}{\partial t}\psi(t) \tag{4.6}$$

for the Hamiltonian (4.5) we obtain

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}(t)e^{\left(-i\gamma_+(t)\right)} & \mathcal{J}(t)e^{\left(-i\gamma_+(t)\right)} \\ \mathcal{J}(t)e^{\left(-i\gamma_-(t)\right)} & \mathcal{F}^*(t)e^{\left(-i\gamma_-(t)\right)} \end{pmatrix} \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \tag{4.7}$$

where

$$\begin{aligned}
\mathcal{F}(t) &= \cos I(t) - i\cos(2\delta(t))\sin I(t), \qquad \mathcal{J}(t) = -i\sin(2\delta(t))\sin I(t), \\
\delta(t) &= \frac{1}{2}\cot^{-1}\theta, \qquad I(t) = \sqrt{\theta^2+1}\int_0^t \lambda(\tau)d\tau, \\
\gamma_{\pm}(t) &= \frac{1}{2}\left[(\omega_1+\omega_2)\,t \pm \phi(t)\right] \qquad (4.8)
\end{aligned}$$

It should be noted that in our calculations we have used the matrix representation for the operators $\hat{K}_i, i = 1,2,3,4$ given by

$$\hat{K}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \hat{K}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \hat{K}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \hat{K}_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \qquad (4.9)$$

With the aid of equation (3.12) the solution of equation (4.6) can be expressed as

$$\exp\left(-\frac{i}{\hbar}\int_0^t \hat{H}(\tau)d\tau\right) = \prod_{i=1}^{4}\exp\left(S_i(t)\hat{K}_i\right), \qquad (4.10)$$

where

$$\begin{aligned}
S_1(t) &= \ln\mathcal{F}(t) - i\gamma_+(t), \qquad S_2(t) = -i\,(\omega_1+\omega_2)\,t, \\
S_3(t) &= \mathcal{F}(t)\mathcal{J}(t), \qquad S_4(t) = \frac{\mathcal{J}(t)}{\mathcal{F}(t)}. \qquad (4.11)
\end{aligned}$$

Using the actions of the one-parameter subgroups $\exp\left(-i\hat{K}_j\right), j = 1,2,3,4$ on functions $f_k(q_1,q_2,t), k = 1,2,3,4$, then we can obtain the most general solution for the wave equation in the Schrödinger picture in the form

$$\begin{aligned}
\psi(q_1,q_2,t) &= \exp\left[-\frac{1}{2}\left(\omega_1 q_1^2 + 2\omega_2 q_2^2\right)\right] \\
&\times \sum_{m,\acute{m},n,r,s,j,k}^{\infty} C_{m,\acute{m},n,r,s,j,k} H_n\left(\sqrt{\omega_1}q_1\right) H_m\left(\sqrt{\omega_2}q_2\right) H_{\acute{m}}\left(\sqrt{\omega_2}q_2\right) \\
&\times \exp\left\{i\left[\sqrt{\frac{\omega_1}{2}}\,(j+k+r+s)\,q_1 + \sqrt{\frac{\omega_2}{2}}\,(j-k-r+s)\,q_2\right]\right\} \\
&\times \exp\left\{-\left[(n-m)\,S_1(t) - \acute{m}S_2(t)\right]\right\}, \\
&\times \exp\left\{-\frac{1}{4}\left[\left(k^2-j^2\right)S_3(t) + \left(r^2-s^2\right)S_4(t)\right]\right\}, \qquad (4.12)
\end{aligned}$$

where $H_n(.)$ stands for the Hermite polynomial and $C_{m,\acute{m},n,r,s,j,k}$ is a constant that depends in general on the frequencies ω_1 and ω_2, such that $C_{m,\acute{m},n,r,s,j,k} = C_{mn}C_{\acute{m}}C_{rs}C_{jk}$.In this case these constants are

$$\begin{aligned}
C_{mn} &= \frac{\sqrt{\omega_1\omega_2}}{\pi n! m!}2^{-(n+m)}\int_{-\infty}^{\infty} f_1(q_1,q_2,0)\, H_n\left(\sqrt{\omega_1}q_1\right) H_m\left(\sqrt{\omega_2}q_2\right) \\
&\quad \exp\left[-\frac{1}{2}\left(\omega_1 q_1^2+\omega_2 q_2^2\right)\right] dq_1 dq_2, \\
C_{\acute{m}} &= \sqrt{\frac{\omega_2}{\pi}}\frac{1}{\acute{m}!2^{\acute{m}}}\int_{-\infty}^{\infty} f_2(0,q_2,0)\, H_{\acute{m}}\left(\sqrt{\omega_2}q_2\right)\exp\left(-\frac{1}{2}\omega_2 q_2^2\right) dq_2, \\
C_{jk} &= \frac{\sqrt{\omega_1\omega_2}}{4\pi^2}\int_{-\pi/\sqrt{2\omega_1}}^{\pi/\sqrt{2\omega_1}}\int_{-\pi/\sqrt{2\omega_2}}^{\pi/\sqrt{2\omega_2}} f_3(q_1,q_2,0)\exp\left[\frac{1}{2}\left(\omega_1 q_1^2-\omega_2 q_2^2\right)\right] \\
&\quad \times\exp\left(i\left[\sqrt{\frac{\omega_1}{2}}(j+k)\,q_1+\sqrt{\frac{\omega_2}{2}}(j-k)\,q_2\right]\right) dq_1 dq_2, \qquad (4.13)\\
C_{rs} &= \frac{\sqrt{\omega_1\omega_2}}{4\pi^2}\int_{-\pi/\sqrt{2\omega_1}}^{\pi/\sqrt{2\omega_1}}\int_{-\pi/\sqrt{2\omega_2}}^{\pi/\sqrt{2\omega_2}} f_4(q_1,q_2,0)\exp\left[\frac{1}{2}\left(\omega_2 q_2^2-\omega_1 q_1^2\right)\right] \\
&\quad \times\exp\left(i\left[\sqrt{\frac{\omega_1}{2}}(r+s)\,q_1+\sqrt{\frac{\omega_2}{2}}(s-r)\,q_2\right]\right) dq_1 dq_2.
\end{aligned}$$

In the previous calculations we have used the Dirac representation for the operators $\hat{a}$ and $\hat{b}$ such that $\hat{a} = (2\omega_1)^{-\frac{1}{2}}(\omega_1\hat{q}_1 + i\hat{p}_1)$ and $\hat{b} = (2\omega_2)^{-\frac{1}{2}}(\omega_2\hat{q}_2 + i\hat{p}_2)$, where $[\hat{q}_i, \hat{p}_j] = \delta_{ij}$.

4.2. **A charged particle in the presence of a magnetic field.** As a second example we shall introduce the charged particle in the presence of a constant magnetic field. The Hamiltonian which represents such a system at exact resonance is given by [11, 21]

(4.14)

$$\hat{H} = \bar{\omega}\left(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}+1\right)+\frac{(\lambda/4)^2}{\omega}\left(\hat{a}^{\dagger 2}+\hat{a}^2+\hat{b}^{\dagger 2}+\hat{b}^2\right)+i(\lambda/2)\left(\hat{a}^\dagger\hat{b}-\hat{a}\hat{b}^\dagger\right),$$

where $\bar{\omega} = \left(\omega + \frac{\lambda^2}{8\omega}\right)$ and λ is the time-independent coupling parameter, while $\hat{a}$ and $\hat{b}$ are Dirce operators.

As before we define four generators $\hat{K}_i, i = 1, 2, 3, 4$ so that

$$\begin{aligned} \hat{K}_1 &= \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1, \qquad \hat{K}_2 = \frac{1}{2}\left(\hat{a}^{\dagger 2} + \hat{b}^{\dagger 2}\right), \\ K_3 &= \frac{1}{2}\left(\hat{a}^2 + \hat{b}^2\right), \qquad K_4 = i\left(\hat{a}^\dagger \hat{b} - \hat{a}\hat{b}^\dagger\right), \end{aligned} \tag{4.15}$$

which satisfy the commutation relations

$$\begin{aligned} \left[\hat{K}_1, \hat{K}_2\right] &= 2\hat{K}_2, \ \left[\hat{K}_1, \hat{K}_3\right] = -2\hat{K}_3, \\ \left[\hat{K}_2, \hat{K}_3\right] &= -\hat{K}_1, \ \text{and} \ \left[\hat{K}_i, \hat{K}_4\right] = 0, \ i = 1, 2, 3, 4, \end{aligned}$$

while the matrix representation is

$$\hat{K}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \hat{K}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \hat{K}_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \hat{K}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.16}$$

Then the matrix image of the partial differential equation (4.6) is

$$i\frac{\partial \psi(t)}{\partial t}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} \bar{\omega} + \lambda/2 & 2\mu \\ -2\mu & -\bar{\omega} + \lambda/2 \end{pmatrix}\psi(t), \tag{4.17}$$

with $\mu = \lambda^2/16\omega$. Solving equation (4.17) we have

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = e^{-i\lambda t/2}\begin{pmatrix} \left(\cos\Omega t - i\frac{\bar{\omega}}{\Omega}\sin\Omega t\right) & -\frac{2i\mu}{\Omega}\sin\Omega t \\ \frac{2i\mu}{\Omega}\sin\Omega t & \left(\cos\Omega t + i\frac{\bar{\omega}}{\Omega}\sin\Omega t\right) \end{pmatrix}\begin{pmatrix} \psi_1(o) \\ \psi_2(o) \end{pmatrix}, \tag{4.18}$$

where $\Omega = \sqrt{\omega^2 + \lambda^2/4}$. In the present case equation (4.10) becomes

$$\exp(-itH) = \exp(S_1(t)K_1)\exp(S_2(t)K_2)\exp(S_3(t)K_3)\exp(S_4(t)K_4), \tag{4.19}$$

where

$$
\begin{aligned}
S_1(t) &= \ln\left(\cos\Omega t + i\frac{\bar{\omega}}{\Omega}\sin\Omega t\right)^{-1}, \\
S_2(t) &= -2i\mu\frac{\sin\Omega t}{\Omega}\left(\cos\Omega t + i\frac{\bar{\omega}}{\Omega}\sin\Omega t\right), \\
S_3(t) &= -2i\mu\frac{\sin\Omega t}{\Omega}\left(\cos\Omega t + i\frac{\bar{\omega}}{\Omega}\sin\Omega t\right)^{-1}, \\
S_4(t) &= -i\lambda t/2.
\end{aligned} \tag{4.20}
$$

The actions of the one-parameter subgroups $\exp(-itK_i)$ as functions $f_i(q_1, q_2, t)$, $i = 1, 2, 3, 4$ leads us to the general solution for the wave function in Schrödingar picture which takes the form

$$
\begin{aligned}
\psi(q_1, q_2, t) &= \sum_{\ell,j,k=0}^{\infty} \sum_{m,n,r,s=-\infty}^{\infty} C_{\ell mnrsjk}\, H_j\left(\sqrt{\omega} q_1\right) H_k\left(\sqrt{\omega} q_2\right) \\
&\quad \times \exp\left[-\frac{\omega}{2}\left(q_1^2 + q_2^2\right)\right] \exp\left[\frac{1}{4}\left(r^2 + s^2\right) S_3(t) + \frac{i}{2}\lambda\ell t\right] \\
&\quad \times \exp\left[-\frac{i}{2}(j + k + 1) S_1(t) + \frac{i}{4}\left(m^2 + n^2\right) S_2(t)\right], \\
&\quad \times \exp\left[i\sqrt{\omega}\left[(r + m) q_1 + (n + s) q_2\right] - i\ell\tan^{-1}\left(\frac{q_2}{q_1}\right)\right]
\end{aligned} \tag{4.21}
$$

where $C_{\ell mnrsjk}$ is a constant, which may be calculated to take the expression

$$C_{\ell mnrsjk} = C_\ell C_{mn} C_{rs} C_{jk},$$

where

$$C_{\ell} = \frac{1}{\pi}\int_{-\infty}^{\infty} f_4(q_1,q_2,0)\exp\left[i\ell\tan^{-1}(q_2/q_1)\right]\exp\left[-\left(q_1^2+q_2^2\right)\right]dq_1dq_2,$$

$$C_{mn} = \frac{\omega}{(2\pi)^2}\int_{-\pi/\sqrt{\omega}}^{\pi/\sqrt{\omega}}\int_{-\pi/\sqrt{\omega}}^{\pi/\sqrt{\omega}} f_2(q_1,q_2,0)\exp\left[-\frac{\omega}{2}\left(q_1^2+q_2^2\right)\right] \times\exp\left[i\sqrt{\omega}\left(mq_1+nq_2\right)\right]dq_1\ dq_2,$$

$$C_{rs} = \frac{\omega}{(2\pi)^2}\int_{-\pi/\sqrt{\omega}}^{\pi/\sqrt{\omega}}\int_{-\pi/\sqrt{\omega}}^{\pi/\sqrt{\omega}} f_3(q_1,q_2,0)\exp\left[\frac{\omega}{2}\left(q_1^2+q_2^2\right)\right] \times\exp\left[i\sqrt{\omega}\left(rq_1+sq_2\right)\right]dq_1\ dq_2, \tag{4.22}$$

$$C_{jk} = \frac{\omega}{\pi}2^{-(j+k)}(j!k!)^{-1}\int_{-\infty}^{\infty} f_1(q_1,q_2,0)H_j\left[\sqrt{\omega}q_1\right]H_k\left[\sqrt{\omega}q_2\right] \times\exp\left[-\frac{\omega}{2}\left(q_1^2+q_2^2\right)\right]dq_1\ dq_2,$$

which complete the required solution. In the following subsection we introduce our third example which represents three coupled modes.

4.3. **Three wave interaction quantum system.** The third example we introduce in this context is the interaction between three electromagnetic fields which describes the Brillouin or Raman scattering process. The system in fact is $\mathbf{SU}(1,1)$ Lie algebra and can be obtained from the interaction between the cavity modes [22, 23, 24]. The Hamiltonian which describes such a system consists of non-degenerate parametric amplifier as well as frequency converter model. The interaction Hamiltonian after removing the time-dependent part is given by

$$\hat{H} = -i\lambda_1\left(\hat{a}\hat{b}-\hat{a}^{\dagger}\hat{b}^{\dagger}\right)-i\lambda_2\left(\hat{a}\hat{c}-\hat{a}^{\dagger}\hat{c}^{\dagger}\right)-i\lambda_3\left(\hat{c}^{\dagger}\hat{b}-\hat{c}\hat{b}^{\dagger}\right), \tag{4.23}$$

where $\lambda_i, i=1,2,3$ are coupling parameters. By defining the generators $\hat{A},\hat{B}$ and $\hat{C}$ such as

$$\hat{A} = \left(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger\right), \qquad \hat{B} = \left(\hat{a}\hat{c} - \hat{a}^\dagger\hat{c}^\dagger\right), \qquad \hat{C} = \left(\hat{c}^\dagger\hat{b} - \hat{c}\hat{b}^\dagger\right), \tag{4.24}$$

which satisfy the commutation relations

$$\left[\hat{A}, \hat{B}\right] = -\hat{C}, \qquad \left[\hat{B}, \hat{C}\right] = \hat{A}, \qquad \left[\hat{C}, \hat{A}\right] = \hat{B}. \tag{4.25}$$

Now let us construct the matrix representation

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \hat{B} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ \hat{C} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.26}$$

then we can write the matrix image of our partial differential equation in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \end{pmatrix} = \begin{pmatrix} 0 & \lambda_3 & \lambda_2 \\ -\lambda_3 & 0 & -\lambda_1 \\ \lambda_2 & -\lambda_1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \end{pmatrix}. \tag{4.27}$$

There are two cases to be considered, the first when $\lambda_3^2 > \lambda_1^2 + \lambda_2^2$, while the second when $\lambda_3^2 < \lambda_1^2 + \lambda_2^2$. For the first case we find that

(4.28)

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1(t) & \mathcal{L}_-^{(3,1,2)}(t) & \mathcal{L}_-^{(2,1,3)}(t) \\ -\mathcal{L}_+^{(3,1,2)}(t) & \mathcal{L}_2(t) & -\mathcal{L}_+^{(2,1,3)}(t) \\ \mathcal{L}_+^{(2,1,3)}(t) & -\mathcal{L}_-^{(1,2,3)}(t) & \mathcal{L}_3(t) \end{pmatrix}\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \\ \psi_3(0) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{L}_i(t) &= \cos\delta t - 2\frac{\lambda_i^2}{\delta^2}\sin^2\left(\frac{\delta t}{2}\right), \qquad i = 1, 2 \\ \mathcal{L}_3(t) &= \cos\delta t + 2\frac{\lambda_3^2}{\delta^2}\sin^2\left(\frac{\delta t}{2}\right), \\ \mathcal{L}_\pm^{(i,j,k)}(t) &= \frac{\lambda_i}{\delta}\sin\delta t \pm 2\frac{\lambda_j\lambda_k}{\delta^2}\sin^2\left(\frac{\delta t}{2}\right), \qquad i, j, k = 1, 2, 3 \end{aligned} \tag{4.29}$$

and $\delta = \sqrt{\lambda_3^2 - \lambda_1^2 - \lambda_2^2}$. Now since the Hamiltonian operator $\hat{H}$ belongs to the lie algebra spanned by the basis elements $\hat{A}, \hat{B}$ and $\hat{C}$, therefore we may write equation (3.12) in the form

$$\exp\left(-it\hat{H}\right) = \exp\left(S_1(t)\hat{A}\right)\exp\left(S_2(t)\hat{B}\right)\exp\left(S_3(t)\hat{C}\right), \tag{4.30}$$

where

$$\begin{aligned} S_1(t) &= -\sinh^{-1}\left[\frac{\mathcal{L}_+^{(2,1,3)}(t)}{\sqrt{\left(\mathcal{L}_-^{(2,1,3)}(t)\right)^2+1}}\right], \\ S_2(t) &= -\sinh^{-1}\left(\mathcal{L}_-^{(2,1,3)}(t)\right), \\ S_3(t) &= -\sin^{-1}\left[\frac{\mathcal{L}_-^{(2,1,3)}(t)}{\sqrt{\left(\mathcal{L}_-^{(2,1,3)}(t)\right)^2+1}}\right]. \end{aligned} \tag{4.31}$$

For the case in which $\lambda_3^2 < \lambda_1^2 + \lambda_2^2$, we have to use the analytic continuation $\delta \longrightarrow i\bar{\delta}$ where $\bar{\delta} = \sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_3^2}$. Following the same analysis as before we can obtain the general solution of equation (4.6) in the present case to take the following expression

$$\begin{aligned} \psi(q_1, q_2, q_3, t) &= \sum_{m,n,r,s,\ell=0}^{\infty} C_{mnrs\ell}\, H_m\left(\frac{q_1+q_2}{\sqrt{2}}\right) \\ &\quad \times H_n\left(\frac{q_1-q_2}{\sqrt{2}}\right) H_r\left(\frac{q_1+q_3}{\sqrt{2}}\right) H_s\left(\frac{q_1-q_3}{\sqrt{2}}\right) \\ &\quad \times \exp\left[-\frac{1}{2}\left(2q_1^2+q_2^2+q_3^2+2i\ell\tan^{-1}\left(\frac{q_2}{q_3}\right)\right)\right] \\ &\quad \times \exp\left(-i\left[(m-n)S_1(t)+(r-s)S_2(t)-i\ell S_3(t)\right]\right), \end{aligned} \tag{4.32}$$

where the normalizing constants $C_{mnrs\ell} = C_{mn}C_{rs}C_{\ell}$ with

$$
\begin{aligned}
C_{mn} &= \frac{1}{\pi} 2^{-(n+m)} (n!m!)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq_1\, dq_2\, \bar{f}(q_1,q_2) \\
&\quad \times H_m\left(\frac{q_1+q_2}{\sqrt{2}}\right) H_n\left(\frac{q_1-q_2}{\sqrt{2}}\right) \exp\left[-\frac{1}{2}\left(q_1^2+q_2^2\right)\right], \\
C_{rs} &= \frac{1}{\pi} 2^{-2(r+s)} (r!s!)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq_1\, dq_3\, \bar{g}(q_1,q_3) \exp\left[-\frac{1}{2}\left(q_1^2+q_3^2\right)\right] \\
&\quad \times H_r\left(\frac{q_1+q_3}{\sqrt{2}}\right) H_s\left(\frac{q_1-q_3}{\sqrt{2}}\right), \qquad (4.33) \\
C_\ell &= \frac{1}{\ell} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq_2\, dq_3\, \bar{h}(q_2,q_3) \exp\left[-\left(q_2^2+q_3^2\right)\right] \\
&\quad \times \exp\left[i\ell \tan^{-1}\left(\frac{q_2}{q_3}\right)\right].
\end{aligned}
$$

As another example of **SU**$(1,1)$ Lie algebra we shall introduce the non-degenerate parametric amplifier which will be considered in the following subsection.

4.4. **Non-degenerate parametric amplifier.** Now we turn our attention to the parametric amplifier system to be our fourth example, that is described by the Hamiltonian [8, 19, 25].

$$
\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 1\right) + \lambda\left(\hat{a}\hat{b} + \hat{a}^\dagger\hat{b}^\dagger\right) \qquad (4.34)
$$

By defining the generators

$$
\hat{K}_+ = \hat{a}^\dagger\hat{b}^\dagger, \quad \hat{K}_- = \hat{a}\hat{b}, \quad \hat{K}_3 = \frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 1\right), \qquad (4.35)
$$

where $\left[\hat{K}_+, \hat{K}_-\right] = -2\hat{K}_3$ and $\left[\hat{K}_3, \hat{K}_\pm\right] = \pm\hat{K}_\pm$, with matrix representation

$$\hat{K}_+ = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \qquad \hat{K}_- = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \qquad \hat{K}_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{4.36}$$

The matrix image of the partial differential equation is now

$$\frac{\partial}{\partial t}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} -i\omega & -\lambda \\ -\lambda & i\omega \end{pmatrix}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} \tag{4.37}$$

with the solution

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} \cos\, kt - i\,(\omega/k)\sin kt & -\,(\lambda/k)\sin kt \\ -\,(\lambda/k)\sin kt & \cos\, kt + i\,(\omega/k)\sin kt \end{pmatrix}\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix}, \tag{4.38}$$

where $k = \sqrt{\omega^2 - \lambda^2}$. Now equation (3.16) can be written as

$$\exp\left(it\hat{H}\right) = \exp\left(S_3(t)\hat{K}_3\right)\ \exp\left(S_2(t)\hat{K}_-\right)\ \exp\left(S_1(t)\hat{K}_+\right). \tag{4.39}$$

The time-dependent complex coefficients $S_1(t), S_2(t)$ and $S_3(t)$ are found to be

$$\begin{aligned} S_1(t) &= \frac{\lambda}{k}\sin\ kt\left(\cos kt - \frac{i\omega}{k}\sin kt\right)^{-1}, \\ S_2(t) &= \frac{\lambda}{k}\sin\ kt\left(\cos kt - \frac{i\omega}{k}\sin kt\right), \\ S_3(t) &= -2\ln\left(\cos kt - \frac{i\omega}{k}\sin kt\right), \end{aligned} \tag{4.40}$$

and the most general solution in this case is given by

$$
\begin{aligned}
\psi\left(q_1, q_2, t\right) &= \sum_{j,k=0}^{\infty} \sum_{s=-\infty}^{\infty} \sum_{m,n=-\infty}^{\infty} C_{mnrsjk} \exp\left[-\frac{\omega}{2}\left(q_1^2+q_2^2\right)\right] \\
&\quad \times \exp\left[-\frac{i}{4}\left(n^2-m^2\right) S_1(t) - \frac{i}{4}\left(s^2-r^2\right) S_2(t)\right] \\
&\quad \times \exp\left(-\frac{1}{2}(j+k+1) S_3(t)\right) H_j\left(\sqrt{\omega} q_1\right) H_k\left(\sqrt{\omega} q_2\right)
\end{aligned} \tag{4.41}
$$

where $C_{mnrsjk} = A_{mn} B_{mnrs} C_{rsjk}$ such that

$$
\begin{aligned}
A_{mn} &= \frac{\omega}{4\pi^2} \int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}} \int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}} \bar{\psi}\left(q_1, q_2\right) \exp\left(-\frac{\omega}{2}\left(q_1^2+q_2^2\right)\right) \\
&\quad \times \exp\left[i\sqrt{\frac{\omega}{2}}\left((m+n)q_1+(m-n)q_2\right)\right] dq_1\, dq_2, \\
B_{mnrs} &= \frac{\omega}{4\pi^2} \int_{-\pi/\sqrt{2\omega}}^{\pi/\sqrt{2\omega}} \bar{\psi}\left(q_1, q_2\right) \exp\left(\omega\left(q_1^2+q_2^2\right)\right) \\
&\quad \times \exp\left(i\sqrt{\frac{\omega}{2}}(m+n+r+s)q_1\right) \\
&\quad \times \exp\left[i\sqrt{\frac{\omega}{2}}(m+r-n-s)q_2\right] dq_1\, dq_2, \\
C_{rsjk} &= \frac{\omega}{\pi} \frac{1}{2^{j+k}} (j!k!)^{-1} \int_{-\infty}^{\infty} \bar{\psi}\left(q_1, q_2\right) \exp\left(-\omega\left(q_1^2+q_2^2\right)\right) \\
&\quad \times \exp\left(i\sqrt{\frac{\omega}{2}}\left((r+s)\, q_1+(r-s)q_2\right)\right) \\
&\quad \times H_j\left(\sqrt{\omega} q_1\right) H_k\left(\sqrt{\omega} q_2\right) dq_1 dq_2
\end{aligned} \tag{4.42}
$$

which complete the solution. The last example we shall introduce is also $\mathbf{SU}(1,1)$ Lie algebra, however, it contains a single mode oscillator. This will be seen in the forthcoming subsection.

4.5. **Degenerate parametric amplifier.** The final example we shall consider is the time-depedent degenerate parametric amplifier model [26, 27, 28, 29]. The model consists of free harmonic oscillator term in addition to the second harmonic generation term which can be described by the Hamiltonian

$$\hat{H}=\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)+\lambda(t)\left\{\hat{a}^{\dagger 2}\exp\left(2i\phi(t)\right)+\hat{a}^{2}\exp\left(-2i\phi(t)\right)\right\} \tag{4.43}$$

As usual we introduce the generators $\hat{K}_{\pm}$ and $\hat{K}_o$ such that $\hat{K}_+=\frac{1}{2}\hat{a}^{\dagger 2}$, $\hat{K}_-=\frac{1}{2}\hat{a}^2$ and $\hat{K}_o=\frac{1}{2}\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)$ .These generators have a matrix representation that takes the form

$$\hat{K}_o=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \hat{K}_-=\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix},\qquad \hat{K}_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \tag{4.44}$$

which satisfy the commutation relation (4.34). The matrix image is now

$$\frac{\partial\psi(t)}{\partial t}=\begin{pmatrix}-i\omega & -2i\ \lambda(t)\ \exp\left(2i\phi(t)\right)\\ 2i\ \lambda(t)\ \exp\left(2i\phi(t)\right) & i\omega\end{pmatrix}\psi(t) \tag{4.45}$$

It is unlikely to find a closed-form solution of these two equations for arbitrary time-dependent coupling $\lambda(t)$ and phase pump $\phi(t)$. However, if we adjust the phase pump $\phi(t)$ to be of the form

$$\phi(t)=-\omega t+2\eta I(t),$$

then the solution of equation (4.45) takes the form

$$\psi(t)=\begin{bmatrix}A\left(t\right) & B^{*}\left(t\right)\\ B\left(t\right) & A^{*}\left(t\right)\end{bmatrix}\psi(0), \tag{4.46}$$

where

$$\begin{aligned}A\left(t\right) &= \left(\cos I(t)-i\frac{\eta}{\sqrt{\eta^2-1}}\sin I(t)\right)e^{i\phi(t)},\\ B\left(t\right) &= i\frac{\sin I(t)}{\sqrt{\eta^2-1}}e^{-i\phi(t)},\end{aligned}$$

and $A^{*}\left(t\right)$, $B^{*}\left(t\right)$ stand for the complex conjugates of $A\left(t\right)$ and $B\left(t\right)$, respectively. Here, $\eta>1$ is an arbitrary constant and $I(t)=\int_0^t\lambda(t')dt$.

Since the Hamiltonian operator $\hat{H}$ belongs to the Lie algebra spanned by $K_o, K_\pm$ which is identical with the Lie algebra of $S\ell(2)$, therefore the operator $\exp\left(-i\int_o^t H(t')dt'\right)$ can be expressed as follows

$$\exp\left(-i\int_o^t H(t')dt'\right) = \exp\left(S_1(t)K_o\right)\,\exp\left(S_2(t)K_-\right)\,\exp\left(S_3(t)K_+\right), \tag{4.47}$$

where the complex coefficients S_1, S_2,and S_3 are given by

$$\begin{aligned}
S_1(t) &= 2\ln\left(\cos I(t) - \frac{i\eta}{\sqrt{\eta^2-1}}\sin I(t)\right) + i\phi(t)\\
S_2(t) &= \frac{-i\sin I(t)}{\sqrt{\eta^2-1}}\left(\cos I(t) - \frac{i\eta}{\sqrt{\eta^2-1}}\sin I(t)\right)\\
S_3(t) &= \frac{-i\sin I(t)}{\sqrt{\eta^2-1}}\left(\cos I(t) - \frac{i\eta}{\sqrt{\eta^2-1}}\sin I(t)\right)^{-1}.
\end{aligned} \tag{4.48}$$

As before we can reach the most general solution in the form

$$\begin{aligned}
\psi(q,t) &= \sum_{n=0}^{\infty}\sum_{m,r=-\infty}^{\infty} A_{mnr}\exp\left[-\frac{\omega}{2}q_1^2 + i\sqrt{\omega}(m+r)q_1\right.\\
&\quad \left. -\frac{1}{4}m^2S_2(t) - \frac{1}{4}r^2S_3(t) + \frac{1}{2}\left(n+\frac{1}{2}\right)S_1(t)\right] H_n\left[\sqrt{\omega}q\right]
\end{aligned} \tag{4.49}$$

where $A_{mnr} = C_mC_nC_r$ represents a constant defined by the expressions

$$\begin{aligned}
C_m &= \frac{\sqrt{\omega}}{2\pi}\int_{-\pi/\sqrt{\omega}}^{\pi/\sqrt{\omega}} f(q)\,\exp\left(-\frac{\omega}{2\hbar}q^2 + im\sqrt{\omega}q\right)dq,\\
C_n &= \sqrt{\frac{\omega}{\hbar\pi}}2^{-n}(n!)^{-1}\int_{-\infty}^{\infty} g(q)\,H_n\left[\sqrt{\omega}q\right]\exp\left(-\frac{\omega}{2}q^2\right)dq,\\
C_r &= \frac{\sqrt{\omega}}{2\pi}\int_{-\pi/\sqrt{\omega}}^{\pi/\sqrt{\omega}} h(q)\,\exp\left(\frac{\omega}{2\hbar}q^2 + ir\sqrt{\omega}q\right)dq.
\end{aligned} \tag{4.50}$$

which complete the solution. Before we close this section we would like to point out that, explicit solutions for some of the mentioned models can be obtained using a direct method for solving the Schrödinger wave function. However, some other time-dependent systems can not be solved. This in fact means that the Lie algebra method is powerful for dealing with a complicated time-dependent quantum systems. As one can see from the previous analysis we can obtain the wave function for the Hamiltonian itself, where the system may be time-dependent or time-independent. This in fact would restrict our result with only one wave function for each model. However, if we think of obtaining the constant of the motion for each model, more precisely the quadratic invariant. In this case we can obtain at least three wave functions for each system two are real and one is complex. Therefore, we have to turn our attention to seek an invariant for the quantum system which has the same properties of the Hamiltonian with the advantages that it is also a constant of the motion. This will be seen in the next section .

5. Quadratic invariants

In this section we introduce in brief the idea of using the quadratic invariant to produce another class of the wave function. To see that let us employ the last example in the previous section to demonstrate such methods. For this reason we introduce the real quadratic constants of the motion which can be constructed to take the form [30, 31]

$$\hat{I} = \beta(t)\hat{p}^2 + \delta(t)\hat{q}^2 + \gamma(t)(\hat{q}\hat{p} + \hat{p}\hat{q}) \tag{5.1}$$

where $\beta(t), \delta(t)$ and $\gamma(t)$ are time-dependent arbitrary functions to be determined. The operators $\hat{p}$ and $\hat{q}$ are the usual momentum and coordinate which satisfy the commutation relation $[\hat{q}, \hat{p}] = i\hbar$. To find these functions the constant of motion should satisfy the relation

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I}, \hat{H}] = 0, \tag{5.2}$$

where $\hat{H}$ is the Hamiltonian of the system. Now let us rewrite the Hamiltonian (4.43) in terms of $\hat{p}$ and $\hat{q}$ using the defination of the Dirca operators given after equation (4.13), thus we have

$$\hat{H}(t) = \frac{1}{2}\nu_{(-)}(t)\hat{p}^2 + \frac{\omega^2}{2}\nu_{(+)}(t)\hat{q}^2 + \mu(t)\left(\hat{p}\hat{q} + \hat{q}\hat{p}\right), \tag{5.3}$$

where

$$\nu_{(\pm)}(t) = \left[1 \pm \frac{2\lambda(t)}{\omega}\cos 2\phi(t)\right] \qquad \text{and} \qquad \mu(t) = \lambda(t)\sin 2\phi(t). \tag{5.4}$$

From equations (5.1), and (5.2) together with equation (5.3) we have

$$\begin{aligned} \frac{d\beta}{dt} + 2\nu_{(-)}(t)\gamma &= 4\mu(t)\beta, \qquad \frac{d\delta}{dt} = 2\omega^2\nu_{(+)}(t)\gamma - 4\mu(t)\delta, \\ \frac{d\gamma}{dt} + \nu_{(-)}(t)\delta &= \omega^2\nu_{(+)}(t)\beta. \end{aligned} \tag{5.5}$$

From these coupled differential equations we are able to construct the invariant. To do so we set $\beta = \rho^2$. Then the first class of quadratic invariant can be written as

$$\hat{I}^{(q)} = \left[\frac{c}{\rho^2} + \left(\frac{2\mu\rho - \dot{\rho}}{\nu_{(-)}(t)}\right)^2\right]\hat{q}^2 + \rho^2\hat{p}^2 + \left(\rho\frac{2\mu\rho - \dot{\rho}}{\nu_{(-)}(t)}\right)(\hat{q}\hat{p} + \hat{p}\hat{q}), \tag{5.6}$$

where $\rho(t)$ is the solution of the nonlinear differential equation

$$\ddot{\rho} - \left(\frac{\dot{\nu}_{(-)}(t)}{\nu_{(-)}(t)}\right)\dot{\rho} + \Gamma^{(-)}(t)\rho = \frac{c}{\rho^3}\nu^2_{(-)}(t) \tag{5.7}$$

and

$$\Gamma^{(-)}(t) = \Omega^2(t) - 2\nu_{(-)}(t)\frac{d}{dt}\left(\frac{\mu(t)}{\nu_{(-)}(t)}\right) \qquad \Omega(t) = \sqrt{\omega^2 - 4\lambda^2(t)} \tag{5.8}$$

On the other hand, if we consider $\delta(t) = \sigma^2(t)$, the second class of quadratic invariant takes the form

$$\hat{I}^{(p)} = \left(\frac{c}{\sigma^2} + \left(\frac{\dot{\sigma} + 2\mu\sigma}{\omega^2\nu_{(+)}(t)}\right)^2\right)\hat{p}^2 + \sigma^2\hat{q}^2 + \left(\sigma\frac{\dot{\sigma} + 2\mu\sigma}{\omega^2\nu_{(+)}(t)}\right)(\hat{q}\hat{p} + \hat{p}\hat{q}), \tag{5.9}$$

where $\sigma(t)$ is the solution of the nonlinear differential equation

$$\ddot{\sigma} - \left(\frac{\dot{\nu}_{(+)}(t)}{\nu_{(+)}(t)}\right)\dot{\sigma} + \Gamma^{(+)}(t)\sigma = \frac{\omega^4}{\sigma^3}\nu^2_{(+)}(t)c \tag{5.10}$$

and

$$\Gamma^{(+)}(t) = \Omega^2(t) + 2\nu_{(+)}(t)\frac{d}{dt}\left(\frac{\mu(t)}{\nu_{(+)}(t)}\right). \tag{5.11}$$

In terms of the creation (annihilation) operators we can rewrite equations (5.6), and (5.9) in the form

$$
\begin{aligned}
\frac{\hat{I}^{(q)}}{\hbar} &= \frac{1}{\omega}\left[\frac{c}{\rho^2}+\omega^2\rho^2+\left(\frac{\dot{\rho}-2\mu\rho}{\nu_{(-)}(t)}\right)^2\right]\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) \\
&+\frac{1}{2\omega}\left(\frac{c}{\rho^2}-\left(\omega\rho+i\frac{\dot{\rho}-2\mu\rho}{\nu_{(-)}(t)}\right)^2\right)\hat{a}^{\dagger 2} \\
&+\frac{1}{2\omega}\left(\frac{c}{\rho^2}-\left(\omega\rho-i\frac{\dot{\rho}-2\mu\rho}{\nu_{(-)}(t)}\right)^2\right)\hat{a}^2.
\end{aligned}
\tag{5.12}
$$

A similar expression can be given for the second class of invariant as

$$
\begin{aligned}
\frac{\hat{I}^{(p)}}{\hbar} &= \omega\left[\frac{c}{\sigma^2}+\frac{\sigma^2}{\omega^2}+\left(\frac{\dot{\sigma}+2\mu\sigma}{\omega^2\nu_{(+)}(t)}\right)^2\right]\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) \\
&-\frac{1}{2\omega}\left(\frac{\omega^2 c}{\sigma^2}-\left(\sigma+i\frac{\dot{\sigma}+2\mu\sigma}{\omega\nu_{(+)}(t)}\right)^2\right)\hat{a}^{\dagger 2} \\
&-\frac{1}{2\omega}\left(\frac{\omega^2 c}{\sigma^2}-\left(\sigma-i\frac{\dot{\sigma}+2\mu\sigma}{\omega\nu_{(+)}(t)}\right)^2\right)\hat{a}^2.
\end{aligned}
\tag{5.13}
$$

We now turn our attention to consider the complex invariant by direct use of equation (1.2). To derive the complex quadratic constant of the motion we define

$$
\hat{I}=\bar{\alpha}(t)\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)+\bar{\beta}(t)\hat{a}^{\dagger 2}+\bar{\beta}^*(t)\hat{a}^2,
\tag{5.14}
$$

where $\bar{\alpha}(t)$ and $\bar{\beta}(t)$ are real and complex time-dependent functions, respectively. If one uses equation (5.2) and equation (5.14) together with equation (4.43), one has

$$
\begin{aligned}
\frac{d}{dt}\bar{\alpha}(t) &= 4i\lambda(t)\left[\bar{\beta}^*(t)\exp(2i\phi(t))-\bar{\beta}(t)\exp(-2i\phi(t))\right] \\
\frac{d}{dt}\bar{\beta}(t) &= 2i\lambda(t)\exp(2i\phi(t))\bar{\alpha}(t)-2i\omega\bar{\beta}(t) \\
\frac{d}{dt}\bar{\beta}^*(t) &= -2i\lambda(t)\exp(-2i\phi(t))\bar{\alpha}(t)+2i\omega\bar{\beta}^*(t).
\end{aligned}
\tag{5.15}
$$

From the above functions and after straightforward calculation we have the following set of solutions:

$$
\begin{aligned}
2\bar{\beta}(t) &= \bar{\beta}(0)\exp\left(2i[\phi(t)-\phi(0)]\right)\left[\left(\frac{\eta^2}{\kappa^2}+1\right)\cos 4\kappa I(t)-\frac{1}{\kappa^2}\right] \\
&\quad +\bar{\beta}^*(0)\exp\left(2i[\phi(t)+\phi(0)]\right)\left[\left(\frac{\eta^2}{\kappa^2}-1\right)\cos 4\kappa I(t)-2i\frac{\eta}{\kappa}\sin 4\kappa I(t)\right. \\
&\quad \left.-\frac{1}{\kappa^2}\right]+\bar{\alpha}(0)\exp\left(2i[\phi(t)]\right)\left[\frac{2\eta}{\kappa^2}\sin^2 2\kappa I(t)+\frac{i}{\kappa}\sin 4\kappa I(t)\right], \qquad (5.16)
\end{aligned}
$$

where $\kappa=\sqrt{\eta^2-1}$. If one uses the above equation together with its complex conjugate, then the expression for the real parameter $\bar{\alpha}(t)$ can be obtained using the relation

$$
|\bar{\beta}(t)|^2=\frac{1}{4}\bar{\alpha}^2(t)+C_1, \qquad (5.17)
$$

where C_1 is constant. As we see from the above equations there is a possibility to produce different classes of the wave function by using different classes of the invariants for the quantum system. In fact these classes of the invariants have the same properties of the Hamiltonian in addition to the flexibility of the time-independence.

6. Conclusion

In the previous sections of the present communication we have introduced in details the general method to find the wave function using Lie algebra approach. Moreover, we have illustrated our method by introducing different examples from the field of quantum mechanics. The most common factor between these models is the closed form of the commutation relation for the Lie algebra generators. In general most of these models are either $\mathbf{SU}(1,1)$ or $\mathbf{SU}(2)$ Lie algebras and this enabled us to apply the method presented here. Also, we have opened the door to seek a wave function for the constants of motion where we have given the degenerate parametric amplifier as an example. Different examples will be introduced in the future contributions. Finally we conclude the following:

i) The method of the solution in the Lie algebra approach is very systematic and the direction towards the solution is indicated by the structure of the Lie algebra, contrary to classical methods where the search for the solution is almost arbitrary and ad hoc.

ii) Through the Lie algebra approach one rediscovers the evolution group which dictates the development of the dynamical system states. Indeed the knowledge of the evolution group is so useful as it gives insight into the symmetrical nature of the dynamical system and its physical properties. Moreover this knowledge of the evolution group provides a technique for constructing the solution of the partial differential equation as it is shown in this paper.

iii) The most general solution computed with the aid of Lie algebraic method is universal, i.e. it unifies seemingly different physical problems, that have essentially the same Lie algebra. The seemingly different problem (that arise from different realizations of the same Lie algebra) look apparently different in the classical approach because the Hamiltonian in this approach does not usually explicitly involve the Lie algebra of the evolution group.

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