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GENERALIZED SUBGROUPS AND HOMOMORPHISMS

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ABSTRACT. In this paper generalized subgroups and homomorphisms of generalized groups are considered. Some results on homomorphisms are proved. One of them is similar to the first isomorphism theorem.

1. INTRODUCTION

In this century mathematicians and physicists have been trying to construct suitable unified theories, for example Twistor theory [2], Isotopies theory [3], and so on. Generalized groups are tools for construction of a unified geometric theory. This notion and its physical background are introduced in [1], where a generalized group is defined as follows [1].

Definition 1.1. A generalized group is a non-empty set G admitting an operation called multiplication, satisfying the following conditions:

i) $(xy)z = x(yz)$ for all x, y, z in G;

ii) For each x in G there exists a unique $e(x)$ in G such that $xe(x) = e(x)x = x$;

iii) For each x in G there exists a x^{-1} in G such that $xx^{-1} = x^{-1}x = e(x)$.

Remark. Generalized groups have the following properties [1]:

i) For each x in a generalized group G there exists a unique x^{-1} in G;

ii) $e(e(x)) = e(x)$ and $e(x^{-1}) = e(x)$ where $x \in G$.

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Definition 1.2. If $e(xy) = e(x)e(y)$ for all $x, y \in G$, then G is called a normal generalized group.

Definition 1.3. Let H be a non-empty subset of a generalized group G such that it is a generalized group with the multiplication inherited from G. Then H is called a generalized subgroup.

Example 1.4. Let $G = \mathbb{R} \times (\mathbb{R} - \{0\})$ where $\mathbb{R}$ is the set of real numbers; then G with the multiplication

$$\begin{array}{rcl} .: G \times G & \longrightarrow & G \\ ((a,b),(c,d)) & \longmapsto & (bc, bd) \end{array}$$

is a normal generalized group and $H = \mathbb{R} \times \{1\}$ is a normal generalized subgroup of it.

Conjecture. It seems that every generalized group is a normal generalized group.

2. SOME RESULTS ON GENERALIZED SUBGROUPS

This section consists of two theorems:

Theorem 2.1. *A non-empty subset H of a generalized group G is a generalized subgroup if and only if for all a and b in H, $ab^{-1} \in H$.*

Proof. If H is a generalized subgroup and $a, b \in H$, then $b^{-1} \in H$ and $ab^{-1} \in H$. Conversely, if $H \neq \emptyset$ and $a, b \in H$ then we have:
$bb^{-1} = e(b) \in H$, $e(b)b^{-1} = b^{-1} \in H$, and $ab = a(b^{-1})^{-1} \in H$.

Corollary 2.2. *Let $\{H_i | i \in I\}$ be a family of generalized subgroups of the generalized group G, and $\bigcap_{i \in I} H_i \neq \emptyset$. Then $\bigcap_{i \in I} H_i$ is a generalized subgroup of G.*

The following example shows that the intersection of two generalized subgroups may be empty.

Example 2.3. The set $G = \mathbb{R} \times (\mathbb{R} - \{0\}) \times \mathbb{R}$ with the multiplication $(a,b,c)(e,f,g) = (be, bf, bg)$, is a normal generalized group.

If $H_1 = \{0\} \times \{1\} \times (\mathbb{R} - \{0\})$ and $H_2 = (\mathbb{R} - \{0\}) \times \{1\} \times \{0\}$ then H_1 and H_2 are generalized subgroups of G and $H_1 \cap H_2 = \emptyset$.

Theorem 2.4. *If G is a generalized group and $a \in G$, then $G_a = \{x \in G : e(x) = e(a)\}$ is a generalized subgroup of G. In fact, G_a is actually a group.*

Proof. For all $b, c \in G_a$ we have $(bc)e(a) = (bc)e(c) = bc$ and $e(a)(bc) = e(b)(bc) = bc$, so $e(bc) = e(a)$. We have seen that $e(c) = e(c^{-1})$. Hence $b, c^{-1} \in G_a$. So $bc^{-1} \in G_a$. Thus G_a is a generalized subgroup of G. Since the identity function is a constant function on G_a, it is also a group.

3. HOMOMORPHISM OF GENERALIZED GROUPS

If G and H are two generalized groups and $f : G \to H$ is a mapping then f is called a homomorphism if $f(ab) = f(a)f(b)$ for all a and b in G.

Example 3.1. Consider two generalized groups $G = \mathbb{R} \times (\mathbb{R} - \{0\})$ and $H = \mathbb{R}$ with the multiplication $(a_1, b_1)(a_2, b_2) = (b_1 a_2, b_1 b_2)$ and $ab = b$ respectively. Then the mapping

$$\begin{array}{rcl} f : G & \longrightarrow & H \\ (a, b) & \longmapsto & \dfrac{a}{b} \end{array}$$

is a homomorphism.

Let $f : G \to H$ be a homomorphism. Then we have the following results:

i) $f(e(a)) = e(f(a))$ is an identity element in H for all $a \in G$;

ii) $f(a^{-1}) = (f(a))^{-1}$, for all $a \in G$;

iii) If K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H;

iv) If D is a generalized subgroup of H and $f^{-1}(D) \neq \phi$, then $f^{-1}(D)$ is a generalized subgroup of G.

v) If G is a normal generalized group, then the set $\{(e(g), f(g)) : g \in$

$G\}$ with the product $(e(a), f(a))(e(b), f(b)) := (e(ab), f(ab))$ is a generalized group, and we denote this generalized group by $\bigcup^{o}_{e(a)\in G} f(G_{e(a)})$.

Theorem 3.2. *Let $a \in G$ and $f : G \to H$ be a homomorphism. Then the kernel of f at a which is denoted by $ker\ f(a) := \{x \in G : f(x) = f(e(a))\}$ is a generalized subgroup of G. Moreover f is one-to-one if and only if $ker\ f(a) = \{e(a)\}$ for all $a \in G$.*

Proof. Let $x, y \in ker\ f(a)$. Then

$$\begin{aligned} f(xy^{-1}) &= f(x)(f(y))^{-1} = f(e(a))(f(e(a)))^{-1} \\ &= f(e(a))f(e(a)^{-1}) = f(e(a))f(e(a)) = f(e(a)). \end{aligned}$$

So $xy^{-1} \in ker\ f(a)$.

Now let f be a one-to-one homomorphism; then $ker\ f(a) = \{e(a)\}$ for all $a \in G$. Conversely if $f(x) = f(y)$, then $f(x)f(y^{-1}) = f(y)f(y^{-1})$. So $f(xy^{-1}) = f(e(y))$. Since $ker\ f(y) = \{e(y)\}$ we have

$$xy^{-1} = e(y) = e(y^{-1}). \tag{1}$$

Similarly $f(y^{-1}x) = f(e(y))$. So

$$y^{-1}x = e(y) = e(y^{-1}). \tag{2}$$

Hence $x = ((y^{-1})^{-1}) = y$ by (1) and (2).

Theorem 3.3. *If G is a group and $f : G \to H$ is a generalized group homomorphism, then $f(G)$ is a generalized subgroup of $H_{f(e)}$, where e is the identity of G. In fact $f(G)$ is actually a group.*

Proof. If $z \in f(G)$, then there exists $a \in G$ such that $f(a) = z$. So $zf(e) = f(a)f(e) = f(ae) = f(a) = z$ and $f(e)z = f(e)f(a) = f(ea) = f(a) = z$. Hence $e(z) = f(e)$. So $z \in H_{f(e)}$.

4. NORMAL GENERALIZED SUBGROUPS; FACTOR GENERALIZED GROUPS

Definition 4.1. A generalized subgroup N of a generalized group G is called a generalized normal subgroup of G if there exists a generalized group E and a homomorphism $f : G \to E$ such that for all $a \in G$ we have $N_a = \emptyset$ or $N_a = ker\ f_a$ where $N_a := N \cap G_a$, $f_a := f|_{G_a}$ and $ker\ f_a = \{x \in G_a : f(x) = f(e(a))\}$.

Example 4.2. Let G be the generalized group of Example 1.4. Then $N = \{(a,b) : a = 2b \text{ or } a = 3b\}$ is a generalized normal subgroup. To show this, let

$$\begin{array}{rcl} f : G & \longrightarrow & G \\ (a,b) & \longmapsto & (\frac{a}{b}, 1). \end{array}$$

Then f is a homomorphism of generalized groups, and satisfies the above conditions.

Theorem 4.3. *Let N be a generalized normal subgroup of the normal generalized group G. Then the set $G/N = \bigcup_{a \in G} G_a/N_a$ is a normal generalized group with the multiplication*

$$\begin{array}{rcl} \cdot : G/N \times G/N & \longrightarrow & G/N \\ (xN_a, yN_b) & \longmapsto & xyN_{ab} \end{array}$$

Sketch of the proof. In the definition of G/N, if $G_a = G_b$, then $N_a = N_b$, so $G_a/N_a = G_b/N_b$. Thus the union defining G/N can be taken over the distinct G_a. We show that the multiplication is well defined. Let $f : G \to E$ be a homomorphism that corresponds to N. Then $N_a = ker\ f_a$ where $a \in G$ and $N_a \neq \emptyset$. If $x_1 N_a = x_2 N_a$ and $y_1 N_b = y_2 N_b$ then there exists $n_a \in N_a$ and $n_b \in N_b$ such that $x_1 = x_2 n_a$ and $y_1 = y_2 n_b$. Because G is a normal generalized group, $e(x_i y_i) = e(x_i) e(y_i) = e(a) e(b) = e(ab)$, for $i = 1, 2$. So $x_1 y_1, x_2 y_2 \in G_{ab}$, and we have

$$\begin{aligned} f_{ab}(x_1 y_1) &= f(x_2 n_a y_2 n_b) = f(x_2) f(n_a) f(y_2) f(n_b) \\ &= f(x_2) f(e(a)) f(y_2) f(e(b)) = f(x_2 e(a)) f(y_2 e(b)) \\ &= f(x_2) f(y_2) = f_{ab}(x_2 y_2). \end{aligned}$$

Hence $x_1 y_1 ker f_{ab} = x_2 y_2 ker f_{ab}$ or $x_1 y_1 N_{ab} = x_2 y_2 N_{ab}$.
We now show that $e(xN_a) = e(x) N_a$. Let $(xN_a)(yN_b) = (yN_b)(xN_a) = xN_a$. Thus $e(b)e(a) = e(ba) = e(a) = e(ab) = e(a)e(b)$, so $e(b) = e(e(a)) = e(a)$ and

$N_a = N_b$. But G_a/N_b is a group with unique identity $e(x)N_a = e(a)N_a$, so $yN_b = yN_a = e(x)N_a$. The proof of the other properties is not difficult.

The generalized group G/N is called the factor generalized group.

Definition 4.4. Let $f : G \to E$ be a generalized group homomorphism. Then $\bigcup_{a\in G} kerf_a$ is called the kernel of f and denoted by $Kerf$.

The next theorem is similar to the first isomorphism theorem.

Theorem 4.5. *Let G be a normal generalized group and $f : G \to E$ be a generalized group homomorphism. Then $G/Ker\ f$ is isomorphic to the $\bigcup^o_{e(a)\in G} f(G_{e(a)})$.*

Proof. We know that for all $a \in G$, G_a is a group, so f_a is a group homomorphism, and $ker\ f_a$ as defined above is just the ordinary kernel of f_a. Therefore, $G_a/ker\ f_a$ is isomorphic to $f(G_a)$, we denote the corresponding isomorphism by g_a. We define $g : G/Ker\ f \longrightarrow \bigcup^o_{e(a)\in G} f(G_{e(a)})$ by $xkerf_x \longmapsto (e(x), g_x(xker\ f_x))$. We now show that g is an isomorphism.
i) g is well defined.
Let $xker\ f_x = yker\ f_y$. Then $G_x = G_y$. So $g_x = g_y$ and $e(x) = e(y)$. Hence $g(xker\ f_x) = g(yker\ f_y)$.
ii) g is a homomorphism.
If $xker\ f_x$ and $yker\ f_y$ belong to $G/ker\ f$, then

$$\begin{aligned} g((xker\ f_x)(yker\ f_y)) &= g((xy)ker\ f_{xy}) \\ &= (e(xy), g_{xy}((xy)ker\ f_{xy})) = (e(xy), f(xy)) \\ &= (e(x)e(y), f(x)f(y)) = (e(x), f(x))(e(y), f(y)) \\ &= g(xker\ f_x)g(yker\ f_y)\ . \end{aligned}$$

iii) g is one-to-one.
If $g(xker\ f_x) = g(yker\ f_y)$, then $(e(x), g_x(xker\ f_x)) = (e(y), g_y(yker\ f_y))$. So $e(x) = e(y)$ and $g_x = g_y$. Since g_x is an isomorphism we have $xker\ f_x = yker\ f_y$.
iv) g is onto.

Let $(e(x), f(x)) \in \bigcup^o_{e(g)} f(G_{e(g)})$ be given. Then $g(xker\ f_x) = (e(x), f(x))$.

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