Arab J. Math. Sc.
Volume 5, Number 1, June 1999
pp. 3-20

UNIQUENESS OF POSITIVE SOLUTION TO A CLASS OF QUASILINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this note we present some uniqueness results for positive solutions of the ordinary differential equation

$$(|u'|^{p-2}u')' + \frac{n-1}{r}|u'|^{p-2}u' + u^{\alpha} + u^{\beta} = 0, u'(0) = u(R) = 0.$$

We show under some conditions on α and β that the differential equation has exactly one non-trivial positive solution.

1. INTRODUCTION

We consider the problem

$$(1.1) \qquad (|u'(r)|^{p-2}u'(r))' + \frac{n-1}{r}|u'(r)|^{p-2}u'(r) + u^{\alpha}(r) + u^{\beta}(r) = 0,$$

for $0 < r < R$, subject to the condition

$$(1.2) \qquad u'(0) = u(R) = 0,$$

where $1 < p < n$, $\beta < \dfrac{np}{n-p} - 1$ and $\alpha \in (\beta, (1+\beta)\dfrac{p}{n} + p - 1)$.

The main goal of this work is to study the uniqueness of positive solutions to (1.1)–(1.2). Our work is motivated by the results of Zhang[**10**],

1991 Mathematics Subject Classification: 34K55-34K65.

who established, for $p = 2$, that problem (1.1)–(1.2) has a unique solution. The method consists of a combination of Kolodner-Coffman arguments [**4, 1**] and of the variational approach.

As in [**1, 10**] define the function ϕ by

$$\phi(r,d) = \frac{\partial u}{\partial d}(r,d) \text{ and } \phi^{'}(r,d) = \frac{\partial u^{'}}{\partial d}(r,d),$$

where u is a solution to (1.1) subject to the condition

$$u(0,d) = d, \quad u^{'}(0,d) = 0, d > 0.$$

Then ϕ satisfies

$$\text{(1.3)} \qquad \begin{cases} (|u'|^{p-2}\phi')' + \dfrac{n-1}{r}|u'|^{p-2}\phi' + \dfrac{\alpha}{p-1}u^{\alpha-1}\phi \\ \qquad +\dfrac{\beta}{p-1}u^{\beta-1}\phi = 0, \\ \phi'(0) = 0 \qquad \phi(0) = 1. \end{cases}$$

An analysis of the zeroes of ϕ proves our main result :

Theorem 1.1. *Let* $\beta \in (p-1, \dfrac{np}{n-p} - 1)$. *Assume that*

$$\text{(1.4)} \qquad \alpha \in (\beta, (1+\beta)\frac{p}{n} + p - 1).$$

Then Problem (1.1)–(1.2) *possesses exactly one non-trivial positive solution.*

In Section 2 we obtain some preliminary results about the solutions of (1.3). In Section 3 we prove the uniqueness of the zero of ϕ and then we get the proof of our main theorem in Section 4.

2. PRELIMINARY RESULTS

Let H be the Sobolev space

$$H(0,1) = \{u \in L^p(0,1); u' \in L^p(0,1),\ u(1) = 0\},$$

endowed with the norm

$$\|u\| = \left[\int_0^1 |u'|^p r^{n-1} dr\right]^{\frac{1}{p}}.$$

For a positive constant t, we introduce the functional I on $H(0,1)$ by

$$I(u) = \frac{1}{p}\int_0^1 |u'|^p r^{n-1} dr - \frac{t}{\alpha+1}\int_0^1 |u|^{\alpha+1} r^{n-1} dr.$$

Set

$$\mu_t = \inf\left\{ I(u); u \in H(0,1),\quad \|u\|_{\beta+1} = 1\right\}. \tag{2.1}$$

By a standard argument we obtain

Lemma 2.1. *For small enough $t > 0$, the infimum in (2.1) is achieved by a positive function say u_t. Moreover $u_t'(r) < 0$ for all $0 < r \leq 1$.*

Proof. Since $\alpha, \beta < p^* - 1$ and

$$\|u\|_{1+\alpha} \leq \|u\|_{p^*}^{\bar{\alpha}} \|u\|_{1+\beta}^{1-\bar{\alpha}},$$

where

$$p^* = np/(n-p), \frac{1}{1+\beta} - \frac{1}{1+\alpha} = \bar{\alpha}\left[\frac{1}{1+\beta} - \frac{1}{p^*}\right],$$

a simple application of the direct method of the calculus of variations shows that the infimum in (2.1) is achieved by some $u_t \geq 0$. Moreover u_t satisfies

$$\begin{cases} (|u'|^{p-2}u')' + \dfrac{n-1}{r}|u'|^{p-2}u' + tu^\alpha + \lambda_t u^\beta = 0, \\ \qquad \text{for } 0 < r < 1, \\ \\ u'(0) = u(1) = 0, \|u\|_{1+\beta} = 1, \end{cases} \tag{2.2}$$

where λ_t is the Lagrange multiplier.

On the other hand we have

$$\begin{aligned}\lambda_t &= \int_0^1 |u_t'|^p r^{n-1} dr - t\int_0^1 (u_t)^{1+\alpha} r^{n-1} dr,\\ &\geq C\|u_t\|_{p^*}^p - t\|u_t\|_{p^*}^{\bar{\alpha}(\alpha+1)}.\end{aligned}$$

We deduce from this that $\lambda_t > 0$ for t small enough and then $u_t > 0$ [**3, 9**]. Now integrate the first equation of (2.2) over $(0, r)$. One sees

$$r^{n-1}|u_t'|^{p-2}u_t'(r) = -\int_0^r [tu^\alpha + \lambda_t u^\beta] s^{n-1} ds.$$

This implies that $u_t'(r) < 0$, for all $0 < r \leq 1$. □

Corollary 2.1. *Define* $T = \{t > 0, \lambda_t > 0\}$ *and* $t_0 = \sup T$. *Then* T *is an interval and* t_0 *is finite.*

Proof. First observe that if $t_1 < t_2$ then $\lambda_{t_1} \geq \lambda_{t_2}$. This implies that T is an interval. Now let $v \in H$ be a function such that $\|v\|_{1+\beta} = 1$. We deduce from (2.1) that

$$(2.3) \qquad \begin{cases} \mu_t \leq \frac{1}{p}\int_0^1 |v'|^p r^{n-1} dr - \dfrac{t}{\alpha+1}\int_0^1 |v|^{\alpha+1} r^{n-1} dr, \\ \mu_t \leq C_0 - tC_1. \end{cases}$$

Next we use (2.2) to obtain

$$\begin{aligned}\lambda_t &= p\mu_t - t\frac{\alpha+1-p}{1+\alpha}\int_0^1 r^{n-1}u_t^{\alpha+1},\\ \lambda_t &\leq C_0 - tC_1,\end{aligned}$$

in view of (2.3). This implies immediately that $t_0 \leq \dfrac{C_0}{C_1}$. □

In addition, let us derive one more results which will be useful in the proof of Theorem 1.1. Hereafter we fix $t \in (0, t_0)$ and λ_t and u_t are denoted by λ and u.

Lemma 2.2. *Let u be a minimizer of* (2.1). *Suppose that there exists $\varphi \in H(0,1)$ such that*

$$\int_0^1 u^\beta \varphi r^{n-1} dr = 0. \tag{2.4}$$

Then

$$\mathcal{J}(\varphi) \geq 0,$$

where

$$\mathcal{J}(\varphi) = \int_0^1 \left(|u'|^{p-2}\varphi'^2 - \frac{\alpha}{p-1} t u^{\alpha-1}\varphi^2 - \frac{\beta}{p-1}\lambda u^{\beta-1}\varphi^2 \right) r^{n-1} dr.$$

Proof. For real numbers σ and s we define

$$F(\sigma, s) = \int_0^1 |u + \sigma u + s\varphi|^{\beta+1} r^{n-1} dr, \tag{2.5}$$

where φ satisfies (2.4). As

$$\int_0^1 u^{\beta+1} r^{n-1} dr = 1, \tag{2.6}$$

we have

$$\begin{cases} F(0,0) & = 1, \\ \dfrac{\partial F}{\partial \sigma}(0,0) & = \beta + 1. \end{cases} \tag{2.7}$$

We deduce from the implicit function theorem that there exists a function σ defined in a small neighbourhood of $(0,0)$ such that

$$\sigma(0) = 0 \text{ and } F(\sigma(s), s) = 1.$$

Define

$$\Gamma(s) = I(u + \sigma(s)u + s\varphi), s \in [-a, a], a > 0.$$

As $\Gamma(0) = I(u)$ is a local minimum of Γ we get

$$\Gamma'(0) = 0 \text{ and } \Gamma''(0) \geq 0.$$

On the other hand an easy computation shows that

$$\begin{aligned} \Gamma''(0) &= (p-1)\int_0^1 (\sigma'(0)u' + \varphi')^2 |u'|^{p-2} r^{n-1} dr \\ &\quad -\alpha t \int_0^1 (\sigma'(0)u + \varphi)^2 u^{\alpha-1} r^{n-1} dr \\ &\quad +\sigma''(0) \int_0^1 (|u'|^p - t u^{\alpha+1}) r^{n-1} dr. \end{aligned} \tag{2.8}$$

The first and second derivation of $F(\sigma(s), s)$ with respect to s give

$$\begin{cases} \displaystyle\int_0^1 (\sigma'(s)u + \varphi)|u + \sigma(s)u + s\varphi|^{\beta-1}(u + \sigma(s)u + s\varphi) r^{n-1} dr = 0, \\ \displaystyle\int_0^1 [\sigma''(s)u + \beta(\sigma'(s)u + \varphi)^2]|u + \sigma(s)u + s\varphi|^{\beta-1} r^{n-1} dr = 0. \end{cases}$$

Therefore

$$\begin{cases} \displaystyle\sigma'(0)\int_0^1 u^{\beta+1} r^{n-1} dr + \int_0^1 u^{\beta}\varphi r^{n-1} dr = 0, \\ \displaystyle\sigma''(0)\int_0^1 x u^{\beta+1} r^{n-1} dr + \beta\int_0^1 u^{\beta-1}\varphi^2 r^{n-1} dr = 0. \end{cases} \tag{2.9}$$

Using (2.4)–(2.6) we infer

$$\sigma'(0) = 0 \text{ and } \sigma''(0) = -\beta \int_0^1 u^{\beta-1}\varphi^2 r^{n-1} dr.$$

Substituting this into (2.8) gives

$$\Gamma''(0) = (p-1)\mathcal{J}(\varphi),$$

which completes the proof. □

Lemma 2.3. *Let u be a positive solution to* (2.2) . *Assume that*

$$\mathcal{J}(\varphi) > 0, \tag{2.10}$$

for all $\varphi \in H(0,1) \setminus \{0\}$ *satisfying* (2.4).

Then the solution w to

$$\begin{cases} (|u'|^{p-2}w')' + \frac{n-1}{r}|u'|^{p-2}w' + \frac{\alpha}{p-1}tu^{\alpha-1}w + \frac{\beta}{p-1}\lambda u^{\beta-1}w = 0 \\ w'(0) = 0 \quad w(0) = 1, \end{cases} \tag{2.11}$$

has exactly one zero point in the interval $(0,1)$. *In particular* $w(1) < 0$.

Proof. First we prove that w vanishes at least once in $(0,1)$. We claim the following Pohozaev type identity:

$$\frac{(\alpha+1-p)}{p-1}t\int_0^1 u^\alpha w r^{n-1}dr + \frac{(\beta+1-p)}{p-1}\lambda\int_0^1 u^\beta w r^{n-1}dr = |u'(1)|^{p-1}u'(1)w(1). \tag{2.12}$$

Indeed from (2.2) and (2.11), we obtain

(2.13)
$$\int_0^1 (|u'|^{p-2}w')'ur^{n-1}dr + \int_0^1 (\frac{\alpha}{p-1}tu^\alpha w + \frac{\beta}{p-1}\lambda u^\beta w)r^{n-1}dr = 0,$$

and

(2.14)
$$\int_0^1 |u'|^{p-2}u'w'r^{n-1}dr - \int_0^1 (\frac{\alpha}{p-1}tu^\alpha w + \frac{\beta}{p-1}\lambda u^\beta w)r^{n-1}dr = 0.$$

Combining (2.13) and (2.14) we get (2.12).

Suppose now that $w(r) > 0$ for all $r \in (0,1)$. The left side of (2.12) is positive while $u'(1) < 0$, then we obtain a contradiction.

Next we assume that there exist $a < b$ such that $w(a) = w(b) = 0$. Without loss of generality we may assume that $w(r) > 0$ for $0 \leq r < a$ and $w(r) < 0$ in (a, b).

Define

$$w_1(r) = \begin{cases} w(r) & \text{if } r < a, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$w_2(r) = \begin{cases} w(r) & \text{if } a < r < b, \\ 0 & \text{otherwise}. \end{cases}$$

Set

$$\varphi = \gamma w_1 + w_2, \gamma \in \mathbb{R}.$$

Since $w_1.w_2 = w_1'.w_2' = 0$, we get

$$\mathcal{J}(\varphi) = \gamma^2 \mathcal{J}(w_1) + J(w_2).$$

On the other hand we deduce from (2.11) that

$$\int_0^1 |u'|^{p-2}(w_i')^2 r^{n-1} dr - \frac{\alpha}{p-1} t \int_0^1 u^{\alpha-1} w_i^2 r^{n-1} dr$$

$$-\frac{\beta}{p-1} \lambda \int_0^1 u^{\beta-1} w_i^2 r^{n-1} dr = 0, \text{ for } i = 1, 2.$$

Therefore $\mathcal{J}(w_i) = 0$, which implies that $\mathcal{J}(\varphi) = 0$, for any γ. Set

$$\gamma = -\int_a^b u^\beta w(r) r^{n-1} dr \Big/ \int_0^a u^\beta w(r) r^{n-1} dr,$$

then φ satisfies (2.4), thus $\mathcal{J}(\varphi) > 0$. We obtain a contradiction. □

Let us note that condition (2.10) is used only to prove that w has at most one zero point in $(0, 1)$. The existence of at least one zero point is given by the Pohozaev type relation (2.12).

We will see later that if $w(1) < 0$ Theorem 1.1 holds trivially. We know, from Lemma 2.3, that this hypothesis is satisfied if $\mathcal{J}(\varphi) > 0$. This is the goal of the next section.

3. NONEXISTENCE RESULTS

It is shown in the previous section that the sign of $\mathcal{J}(\varphi)$ is crucial. In this section we shall prove that $\mathcal{J}(\varphi) > 0$ for all $\varphi \in H(0,1) \setminus \{0\}$ such that

$$\int_0^1 u^\beta \varphi r^{n-1} dr = 0. \tag{3.1}$$

To this end we follow an idea used in **[8, 10]**.

Let

$$J \overset{\text{def}}{=} \inf\left\{\mathcal{J}(\varphi), \varphi \in H, \int_0^1 u^\beta \varphi r^{n-1} dr = 0, \int_0^1 u^{\beta-1}\varphi^2 r^{n-1} dr = 1\right\}. \tag{3.2}$$

Observe that from Lemma 2.3 $J \geq 0$ and that by a standard method of the calculus of variations, the infimum in (3.2) is achieved. Let w be a minimizer of (3.2) and assume that $J = 0$. Then w satisfies the following

$$\begin{cases} (r^{n-1}|u'|^{p-2}w')' + r^{n-1}\{\frac{\alpha}{p-1} t u^{\alpha-1} w + \frac{\beta}{p-1}\lambda u^{\beta-1} w\} \\ = \dfrac{\alpha+1-p}{p-1} t r^{n-1} u^\beta \displaystyle\int_0^1 u^\alpha(s) w(s) s^{n-1} ds \\ w'(0) = w(1) = 0, \displaystyle\int_0^1 u^\beta w r^{n-1} dr = 0, \int_0^1 u^{\beta-1} w^2 r^{n-1} dr = 1. \end{cases} \tag{3.3}$$

Set

$$\mathcal{P} = \{r : r \in [0,1], w(r) > 0\},$$

$$\mathcal{N} = \{r : r \in [0,1], w(r) < 0\}.$$

We have

Proposition 3.1. *Let u be a positive solution to* (2.2) *and let w be a solution of (3.3). Then*

$$\int_0^1 u^\alpha w r^{n-1} dr \neq 0. \tag{3.4}$$

Moreover $\mathcal{P}$ and $\mathcal{N}$ cannot simultaneously have more than one open interval, and if

$$\int_0^1 u^\beta(r) w(r) r^{n-1} dr > 0,$$

then $\mathcal{P} = (a, b)$ for some $a, b \in (0, 1)$.

Proof. We argue by contradiction, and assume that

$$\int_0^1 u^\alpha w r^{n-1} dr = 0. \tag{3.5}$$

We note in this case that there exists at least $a_0 \in (0, 1)$ such that $w(a_0) = 0$.

Suppose first that w has exactly one zero point. Since $-w$ is also a solution of (3.3) we can suppose that $w > 0$ in $(0, a_0)$. Let

$$v = (u^{\alpha-\beta}(r) - u^{\alpha-\beta}(a_0)) w(r), \text{ for } r \in (0, 1).$$

Since u is decreasing, it is easy to see that

$$v(r) > 0 \text{ for all } r \neq a_0.$$

In particular

$$L = \int_0^1 u^\beta(r) v(r) r^{n-1} dr > 0,$$

but

$$\begin{aligned} L &= \int_0^1 u^\alpha(r) w(r) r^{n-1} dr - u^{\alpha-\beta}(a_0) \int_0^1 u^\beta(r) w(r) r^{n-1} dr \\ &= 0. \end{aligned}$$

This is a contradiction.

It follows from the above result that if (3.5) holds then w has at least two zero points in $(0,1)$. Let a_0, a_1 be the first two zeroes of w. We may assume that $w > 0$ in $(0, a_0)$ and $w < 0$ in (a_0, a_1).

As in the proof of Lemma 2.2 define

$$w_1(r) = \begin{cases} w(r) & 0 < r < a_0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$w_2(r) = \begin{cases} w(r) & a_0 < r < a_1, \\ 0 & \text{otherwise.} \end{cases}$$

For real numbers γ_1 and γ_2 we set

$$\bar{w} = \gamma_1 w_1 + \gamma_2 w_2.$$

We deduce from (3.3) that

$$\int_0^1 ((r^{n-1}|u'|^{p-2}\bar{w}')'\bar{w} \quad + \quad r^{n-1}(\tfrac{\alpha}{p-1} t u^{\alpha-1}\bar{w}^2 + \tfrac{\beta}{p-1}\lambda u^{\beta-1}\bar{w}^2))dr$$

$$= \quad t\frac{\alpha+p-1}{p-1}\int_0^1 u^\alpha \bar{w} r^{n-1} dr \int_0^1 u^\beta \bar{w} r^{n-1} dr.$$

Now we choose γ_1 and γ_2 such that

$$\int_0^1 u^\beta \bar{w} r^{n-1} dr = 0, \quad \int_0^1 u^{\beta-1}\bar{w}^2 r^{n-1} dr = 1.$$

We conclude that $\bar{w}$ is also a minimizer of (3.2). Since $\bar{w}$ satisfies

$$(|u'|^{p-2}w')' + \frac{n-1}{r}|u'|^{p-2}w' + \frac{\alpha}{p-1} t u^{\alpha-1} w + \frac{\beta}{p-1}\lambda u^{\beta-1} w = 0, \quad 0 < r < a_0, \tag{3.6}$$

$\bar{w}$ satisfies (3.6) for all r in $(0,1)$. But $\bar{w} \equiv 0$ in $(a_1, 1)$ and this is impossible, then (3.4) holds. Therefore $\bar{w}$ satisfies (3.6) in the whole interval $(0,1)$. But $\bar{w}(r) \equiv 0$ for $b < r < 1$. So this is impossible. Now suppose, without loss of generality, that

$$\int_0^1 u^\alpha w r^{n-1} dr < 0,$$

and that there exist $a_1 < b_1 < a_2 < b_2$ such that $(a_i, b_i) \subset \mathcal{P}, w(a_i) = w(b_i) = 0, \quad i = 1, 2$.

Set

$$w_i(r) = \begin{cases} w(r) & a_i < r < b_i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2$.

Let $\varphi = \gamma w_1 + w_2$ where γ is given such that

$$\int_0^1 u^\beta \varphi r^{n-1} dr = 0.$$

By Lemma 2.3 we deduce $\mathcal{J}(\varphi) \geq 0$ and by (3.3) we obtain

$$\begin{aligned} J(\varphi) &= -\tfrac{\alpha+1-p}{p-1} t \int_0^1 u^\alpha w r^{n-1} dr \\ &\times \left\{ \gamma^2 \int_{a_1}^{b_1} u^\beta w_1 r^{n-1} dr + \int_{a_2}^{b_2} u^\beta w_2 r^{n-1} dr \right\} < 0. \end{aligned}$$

Which gives a contradiction and ends the proof. □

By a similar argument we can prove that if

$$\int_0^1 u^\alpha w r^{n-1} dr < 0,$$

then $\mathcal{N} = (a, b)$ for some $a, b \in (0, 1)$.

Now we are in position to prove our crucial theorem.

Theorem 3.2. *Problem* (3.2) *has no solution and then* $\mathcal{J} > 0$.

Proof. The arguments to follow, for proving the above theorem, are based on a Pohozaev type identity and Proposition 3.1. The proof consists of three steps. Assume that w is a solution to (3.2).

Step1. We claim that

$$|u'(1)|^{p-2}u'(1)w'(1) = \frac{1}{p-1}\left(p - \frac{n(\alpha+1-p)}{\beta+1}\right)t\int_0^1 u^\alpha w r^{n-1}dr. \tag{3.7}$$

Indeed, let $v(r) = ru'(r)$. A direct computation shows that

$$\begin{aligned}(r^{n-1}|u'|^{p-2}v')' &= -\frac{p}{p-1}r^{n-1}(tu^\alpha + \lambda u^\beta)\\ &\quad -\frac{r^{n-1}}{p-1}(\alpha t u^{\alpha-1} + \beta\lambda u^{\beta-1})v,\end{aligned} \tag{3.8}$$

and then

$$\begin{aligned}\int_0^1 (r^{n-1}|u'|^{p-2}v')'wdr &= -\frac{1}{p-1}\int_0^1 r^{n-1}(\alpha t u^{\alpha-1} + \beta\lambda u^{\beta-1})wvdr\\ &\quad -\frac{p}{p-1}t\int_0^1 r^{n-1}u^\alpha wdr,\end{aligned} \tag{3.9}$$

since

$$\int_0^1 r^{n-1}u^\beta wdr = 0.$$

Next we use (3.2) to obtain

$$\begin{aligned}\int_0^1 w(r^{n-1}|u'|^{p-2}v')'dr &= -w'(1)|u'(1)|^{p-1}u'(1)\\ &\quad -\frac{1}{p-1}\int_0^1 r^{n-1}(\alpha t u^{\alpha-1}w + \beta\lambda u^{\beta-1}w)vdr\\ &\quad +\frac{\alpha+1-p}{p-1}t\int_0^1 r^{n-1}u^\beta vdr\int_0^1 r^{n-1}u^\alpha wdr.\end{aligned} \tag{3.10}$$

Combining (3.9)–(3.10) and using

$$\int_0^1 u^{1+\alpha}r^{n-1}dr = 1 \text{ and } \int_0^1 u^\alpha v r^{n-1} = \frac{1}{\alpha+1},$$

we obtain (3.7).

Step 2. We have from proposition 3.1 that $\int_0^1 u^\alpha w r^{n-1} dr \neq 0$ and we may assume, without loss of generality, that $\int_0^1 u^\alpha w r^{n-1} dr < 0$, then $\mathcal{N} = (a, b)$. Suppose that $0 < a < b < 1$. Now as $u'(1).w'(1) > 0$ and $(1+\alpha) < (1+\beta)\frac{p}{n} + p$ we get a contradiction from (3.7).

Step 3. We deduce from Step 2 that w has exactly one sign change point, that is $\mathcal{N} = (0, a)$ or $\mathcal{N} = (a, 1)$. Therefore

$$\int_0^1 [u^\alpha(r) - u^{\alpha-\beta}(a)u^\beta(r)]w(r)r^{n-1}dr > 0,$$

hence

$$\int_0^1 u^\alpha w r^{n-1} dr > 0.$$

We use (3.7) again to get a contradiction. □

4. PROOF OF THEOREM 1.1

Instead of proving Theorem 1.1, we relate (1.1) to problem

$$(4.1) \qquad \begin{cases} (|u_t'|^{p-2}u_t')' + \frac{n-1}{r}|u_t'|^{p-2}u_t' + tu_t^\alpha + \lambda_t u_t^\beta = 0 \\ u_t'(0) = u_t(1) = 0. \end{cases}$$

Define v by

$$(4.2) \qquad u_t(r) = (\frac{\lambda_t}{t})^{\frac{1}{\alpha-\beta}} v(l_t r),$$

where

$$l_t = t^{\frac{p-\beta-1}{p(\alpha-\beta)}} \lambda_t^{\frac{\alpha+1-p}{p(\alpha-\beta)}}.$$

Then

$$(4.3)\qquad \begin{cases} (|v'|^{p-2}v')' + \frac{n-1}{r}|v'|^{p-2}v' + v^{\alpha} + v^{\beta} = 0 \\ v'(0) = v(l_t) = 0. \end{cases}$$

From Corollary 2.1 we deduce the following

Lemma 4.4.

$$l_t \longrightarrow +\infty \quad \textit{as} \quad t \longrightarrow 0 \qquad \textit{and} \qquad l_t \longrightarrow 0 \quad \textit{as} \quad t \longrightarrow t_0.$$

Proof of Theorem 1.1. For $d > 0$, let $u(\cdot, d)$ be a solution of the following problem

$$(4.4)\qquad (|u'|^{p-2}u')' + \frac{n-1}{r}|u'|^{p-2}u' + |u|^{\alpha-1}u + |u|^{\beta-1}u = 0,$$

supplemented with the initial conditions

$$(4.5)\qquad u'(0) = 0, \quad u(0) = d > 0.$$

It is known, from Pohozaev identity [3], that Problem (4.4)–(4.5) has no positive solution on $(0, \infty)$. Then there exists $r(d) > 0$ such that $u(r, d) > 0$ for $r \in [0, r(d)[$, $u(r(d), d) = 0$ and, by Lemma 2.1, $u'(r, d) < 0$ for all $0 < r \leq r(d)$. Let $t \in (0, t_0)$ such that

$$(4.6)\qquad r(d) = l_t.$$

Set

$$(4.7)\qquad u_t(r) = (\frac{\lambda_t}{t})^{\frac{1}{\alpha-\beta}} u(rl_t, d).$$

Then

$$(4.8)\qquad (|u_t'|^{p-2}u_t')' + \frac{n-1}{r}|u_t'|^{p-2}u_t' + tu_t^{\alpha} + \lambda_t u_t^{\beta} = 0,$$

and

$$u_t'(0) = u_t(1) = 0. \tag{4.9}$$

Define for $r \in (0,1)$,

$$w_t(r) = \frac{\partial u_t}{\partial d} \text{ and } w_t'(r) = \frac{\partial u_t'}{\partial d}.$$

Hence w_t satisfies

$$\begin{cases} (|u_t'|^{p-2}w')' + \frac{n-1}{r}|u_t'|^{p-2}w' + \frac{\alpha}{p-1}tu_t^{\alpha-1}w + \frac{\beta}{p-1}\lambda_t u_t^{\beta-1}w = 0 \\ w'(0) = 0 \qquad w(0) = 1 \end{cases} \tag{4.10}$$

And then, by (2.3) and Proposition 3.1, $w_t(1) < 0$.

We deduce from this that the function $w(r) = w_t(\frac{r}{l_t})$ satisfies

(4.11)

$$\begin{cases} (|u'|^{p-2}w')' + \frac{n-1}{r}|u'|^{p-2}w' + \dfrac{\alpha}{p-1}tu^{\alpha-1}w + \dfrac{\beta}{p-1}\lambda_t u^{\beta-1}w = 0, \\ w'(0) = 0 \qquad w(0) = 1, \end{cases}$$

moreover $w(l_t) = w_t(1) < 0$.

On the other hand, since $u(r(d), d) = 0$, we get

$$u_r'(r(d), d)r'(d) + w(r(d), d) = 0.$$

Then we deduce $r'(d) < 0$ for all $d > 0$. Therefore Problem (1.1)–(1.2) has exactly one positive solution. □

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Date received December 21, 1998.