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AUTOMORPHISMS OF P_0P-LATTICES

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ABSTRACT. P_0-lattices, which are P-algebras, are called P_0P-lattices. The chain base in a P_0-lattice is not necessarily unique. The automorphisms of a finite P_0P-lattice $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ that make the chain of constants $e_0, \cdots, e_{n-1}$ fixed, where the chain satisfies the condition: If a is an atom of B and $0 \neq e_i$ then $0 \neq ae_i$, for all $1 \leq j \leq i \leq n-2$, were described. In this paper we describc the automorphisms of a finite P_0P-lattice $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ that make the chain of constants $e_0, \cdots, e_{n-1}$ fixed, where $e_0, \cdots, e_{n-1}$ is any chain base for P, and we obtain the above results as corollaries.

1. PRELIMINARIES

We adopt the notation of [3] here. A *P_0-lattice* is a bounded distributive lattice P which is generated by its center B of all complemented elements of P and a finite sub-chain $0 = e_0 < e_1 < \cdots < e_{n-1}$ containing 0 and 1. It is denoted by $P = \langle B, e_0, \cdots, e_{n-1}\rangle$. Furthermore

(i) $e_0, \cdots, e_{n-1}$ is called a *chain base* of P

(ii) A P_0-lattice P is of *order* n $(n > 2)$ if n is the smallest integer such that P has a chain base with n-terms

(iii) Every element $x \in P$ can be written in the form: $x = d_1e_1 \vee \cdots \vee d_{n-1}e_{n-1} = \bigvee_{i=1}^{n-1} d_ie_i$, where $d_i \in B, i = 1, \cdots, n-1$ and $d_1 \geq d_2 \geq \cdots \geq d_{n-1}$. Such a representation is called a *monotonic representation* (*mon.rep.*) of x.

Let P be a bounded distributive lattice with center B. Let the largest element $z \in P$ $(z \in B)$ such that $zx \leq y$, if it exists, be denoted by $x \longrightarrow y$ $(x \Rightarrow y)$. Let $\neg x = x \longrightarrow 0$. and $!x = (1 \Rightarrow x)$. If $x \longrightarrow y$ $(x \Rightarrow y)$ exists for all $x, y \in P$ then P is called a *Heyting algebra* (*a B-algebra*). It is clear that $x \Rightarrow y = !(x \longrightarrow y)$. An element $x \in P$ is called dense if $\neg x = 0$. A *P-algebra* is a B-algebra satisfying $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ for all $x, y \in P$.

If P is a P-algebra [1], then:

$$!(xy) = !x!y forall x, y \in P,$$
$$x \longrightarrow (y \vee z) = (x \longrightarrow y) \vee (x \longrightarrow y),$$
$$yz \longrightarrow x = (y \longrightarrow x) \vee (z \longrightarrow x),$$
$$x \longrightarrow yz = (x \longrightarrow y)(x \longrightarrow z)\, and\, (y \vee z) \longrightarrow x = (y \longrightarrow x)(z \longrightarrow x).$$

P_0-lattices which are P-algebras are called *P_0P-lattices* [3]. $a \in P$ is Boolean if a has a complement. Let the least Boolean element greater than or equal to x (if it exists) be denoted by $x!$. It was proved in [3] that in a P_0P-lattice $P = \langle B, e_0, \cdots, e_{n-1} \rangle$,

(1) $(x \Rightarrow y)(y \Rightarrow z) \leq (x \Rightarrow z)$

(2) $x! = \overline{x \Rightarrow 0}$, ($\overline{x \Rightarrow 0}$ is the complement of $x \Rightarrow 0$),

(3) $(x \vee y)! = x! \vee y!$,

(4) Everyelement $x \in P$ can be written in the form $x = \bigvee_{i=1}^{n-1} D_i(x)\, e_i$, where $D_i(x) = x!(e_i \Rightarrow x), i = 1, \cdots, n-1$ and the following properties hold:

(a) $D_1(x) \geq \cdots \geq D_{n-1}(x)$,

(b) $D_i(x \vee y) = D_i(x) \vee D_i(y)$,

(c) $D_i(xy) = D_i(x)\, D_i(y)$,

(d) $D_i(b) = b for b \in B$,

(e) $D_i(e_j) = e_j!$ for $i \leq j$ and $D_i(e_j) = e_j!(e_i \Rightarrow e_j)$ for $i > j$ and in particular $D_{n-1}(e_j) = !e_j$.

2. P_0P-LATTICE HOMOMOPHISMS

Each P_0P-lattice $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ used in this section is finite and the center B is a Boolean algebra.

Definition 2.1. Let $P = \left\langle B, e_{0,\cdots,e_{n-1}}\right\rangle$ and $P' = \left\langle B', e_0', \cdots, e_{m-1}'\right\rangle$ be two P_0-lattices of orders n and m respectively, $m \geq n$, then a lattice homomorphism h of P into P' is called a P_0-*homomorphism*, provided,
(i) $h|_B : B \to B'$ is a Boolean homomorphism.
(ii) $h(e_1), \cdots, h(e_{n-2})$ is chain in $P' - B'$

If P and P' are P_0P-lattices, then h is said to be a P_0P-homomorphism provided that: $h(x \Rightarrow y) = h(x) \Rightarrow h(y)$.

A one-to-one P_0P-homomorphism of a P_0P-lattice P onto itself is called a P_0P-automorphism. Hence, if h is an automorphism of a P_0P-lattice $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ then $h(B) = B$ and $h(e_0), \cdots, h(e_{n-1})$ is a chain base of P.

Definition 2.2. Let $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ be a P_0P-lattice of order n. Let
(i) $Fe_i = \{a : a \text{ is an atom of } B \text{ and } ae_i = 0\}$, $i = 1, \cdots, n-1$.
(ii) $Pe_i = \{a : a \text{ is an atom of } B \text{ and } 0 < ae_i < a \text{ and } ae_j = 0 \text{ for all } j > i\}$, $i = 1, \cdots, n-1$.
(iii) $Pe_{ik} = \{a : a \in Pe_i \text{ and } ae_k < a\}$, $k = i, \cdots, n-1$,
(iv) $P_{e_{ik}}^{s(j)} = \{a : a \in Pe_{ik} \text{ and } ae_{t-1} = ae_t \text{ iff } t \in s(j), s(j) \subseteq \{i+1, \cdots, k\}\}$, where $s : \{1, 2, \cdots, 2^{k-i}\} \to 2^{\{i+1,\cdots,k\}}$ and $2^{\{i+1,\cdots,k\}}$ is the power set of $\{i+1, \cdots, k\}$, $\left(\text{if } k = 1, \text{ then } s(j) = \phi \text{ and } Pe_{ii}^{\phi} = Pe_{ii} = Pe_i\right)$.
(v) $Se_i = Fe_{i-1} - \left(Fe_i \bigcup Pe_i\right)$, $i = 1, \cdots, n-1$.

If A is the set of all atoms in B, then it is clear that

(f_1) $A = Fe_0 \supseteq Fe_1 \supseteq \cdots \supseteq Fe_{n-1} = \phi$,

(f_2) $Pe_i \subseteq Fe_{i-1}$, $i = 1, \cdots, n-2$.

(f_3) $Se_{n-1} = Fe_{n-2} - \left(Fe_{n-1} \bigcup Pe_{n-1}\right) = Fe_{n-2} - \phi = Fe_{n-2}$.

Lemma 2.1. *$Pe_i \bigcap Pe_j = \phi$ for all $1 \leq i \neq j \leq n-1$.*

Proof. Let $a \in Pe_i \bigcap Pe_j$, $i < j$. Since $a \in Pe_j$ then $0 \neq ae_j < a$ and $ae_{j'} = 0$ for $j' < j$. Since $i < j$ then $ae_i = 0$, which contradicts the fact that $a \in Pe_i$. Hence $Pe_i \bigcap Pe_j = \phi$. □

Lemma 2.2. $Se_i \bigcap Se_j = \phi$ *for all* $1 \leq i \neq j \leq n-1$.

Proof. Suppose not and let $a \in Se_i \bigcap Se_j, i < j$. then $a \in Se_i$. Hence $a \in Fe_{i-1}$ and $a \notin Fe_i$. Now, $a \in Se_j$ implies $a \in Fe_{j-1}$ and $a \notin Fe_j$. Since $i < j$, then $j-1 > i$. Hence, $Fe_{j-1} \subseteq Fe_j$. Since $a \in Fe_{j-1}$ then $a \in Fe_i$. Contradiction. □

Lemma 2.3. $P^{s(j)}_{e_{ik}} \bigcap P^{s(j')}_{e_{ik}} = \phi$ *for* $i \leq j \neq j' \leq 2^{k-i}$.

Proof. Since s is one-to-one, $j \neq j'$ implies $s(j) \neq s(j')$. Hence there exists i_0, $i < i_0 \leq k$, such that $i_0 \in s(j)$ and $i_0 \notin s(j')$ or $i_0 \notin s(j)$ and $i_0 \in s(j')$. In the first case, let: $a \in P^{s(j)}_{e_{ik}} \bigcap P^{s(j')}_{e_{ik}}$. Then

(i) $a \in P^{s(j)}_{e_{ik}}$ and $i_0 \in s(j)$ implies $ae_{i_0-1} = ae_{i_0}$, and
(ii) $a \in P^{s(j')}_{e_{ik}}$ and $i_0 \notin s(j')$ implies $ae_{i_0-1} < ae_{i_0}$which contradicts (i). We arrive at the same contradiction in the second case. Hence, $P^{s(j)}_{e_{ik}} \bigcap P^{s(j')}_{e_{ik}} = \phi$. □

Lemma 2.4. $P_{e_{ik}} = \bigcup\limits_{j=1}^{2^{k-i}} P^{s(j)}_{e_{ik}}$, *where some of the terms may be empty.*

Proof. Let $a \in Pe_{ik}$, then either

(i) $ae_{i(l-1)} \neq ae_{il}$ for all $i+1 \leq l \leq k,\ k > i$, i.e. $ae_{ii} \neq ae_{i(i+1)} \neq \cdots \neq ae_{ik}$; hence, $a \in P^{s(j)}_{e_{ik}}$ where $s(j) = \phi$, or

(ii) $ae_{i(l_1-1)} = ae_{il_1}, \cdots, ae_{i(l_{i_0}-1)} = ae_{il_{i_0}}$ for some $i+1 \leq l_1, \cdots, l_{i_0} \leq k$. Let $s(j) = \{l_1, \cdots, l_{i_0}\}$. Then $s(j) \subseteq \{i+1, \cdots, k\}$ and $l \in s(j) \leftrightarrow ae_{i(l-1)} = ae_{il}$. Hence $a \in P^{s(j)}_{e_{ik}}$ which implies $P_{e_{ik}} \subseteq \bigcup\limits_{j=1}^{2^{k-i}} P^{s(j)}_{e_{ik}}$. The converse inclusion is obvious from the definition. □

Remark 2.1.

(i) Using Lemmas 2.5 and 2.6 it is clear that Pe_{ik} has the partition

$$P_{e_{ik}} = \bigcup_{j=1}^{2^{k-i}} P^{s(j)}_{e_{ik}}, \text{ (some of the terms may be empty)}$$

(ii) $Pe_{ik} - Pe_{i(k-1)}$, has the partition

$$Pe_{ik} - Pe_{i(k+1)} = \left(\bigcup_{j=1}^{2^{k-i}} P_{e_{ik}}^{s(j)}\right) - Pe_{i(k+1)} =$$

$$\bigcup_{j=1}^{2^{k-i}} \left(P_{e_{ik}}^{s(j)} - Pe_{i(k+1)}\right), \text{ (some of the terms may be empty)}$$

(iii) $$Pe_i = Pe_{ii}$$
$$= \left(Pe_{ii} - Pe_{i(i+1)}\right) \bigcup \left(Pe_{i(i+1)} - Pe_{i(i+2)}\right) \bigcup \cdots \bigcup \left(Pe_{i(n-2)} - Pe_{i(n-1)}\right)$$
$$= \bigcup_{k=i}^{n-2} \left(Pe_{ik} - Pe_{i(k+1)}\right)$$

Hence, Pe_i has the partition

$$Pe_i = \bigcup_{k=i}^{n-2} \left(Pe_{ik} - Pe_{i(k+1)}\right) = \bigcup_{k=i}^{n-2} \bigcup_{j=i}^{2^{k-i}} \left(P_{e_{ik}}^{s(j)} - Pe_{i(k+1)}\right)$$

(some of the terms may be empty)

(iv) $Fe_1 = (Fe_1 - Fe_2) \cup (Fe_2 - Fe_3) \cup \cdots \cup (Fe_{n-2} - Fe_{n-1})$

$$= \bigcup_{i=1}^{n-2} (Fe_i - Fe_{i+1}).$$

Now, $Se_i = Fe_{i-1} - (Fe_i \cup Pe_i)\,, Fe_i, Pe_i \subseteq Fe_{i-1}$ and $Fe_i \cap Pe_i = \phi$, hence $Se_i \cup Pe_i = Fe_{i-1} - Fe_i$, $i = 1, \cdots, n-1$ and (iv) implies

(v) $Fe_1 = \bigcup_{i=1}^{n-2} (Fe_i - Fe_{i+1}) = \bigcup_{i=2}^{n-1} (Fe_{i-1} - Fe_i) = \bigcup_{i=2}^{n-2} (Fe_i - Se_i)$, where some of the term may be empty.

Hence, if A is the set of all atoms in the center B of a P_0P-lattice $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ of order n, then using Definition 2.2, A has a partition $A = Fe_1 \cup Pe_1 \cup Se_1$. Hence, using Remark 2.1 (v), $A = Fe_1 \cup Pe_1 \cup Se_1 = \bigcup_{i=2}^{n-1} (Pe_i \cup Se_i) \cup Pe_1 \cup Se_1 = \bigcup_{i=1}^{n-1} (Pe_i \cup Se_i)$, and using Remark 2.1 (iii), A has the partition

(vi) $$A = \bigcup_{i=1}^{n-2} \left[\bigcup_{k=i}^{n-2} \left(\bigcup_{j=1}^{2^{k-i}} \left(P_{e_{ik}}^{s(j)} - Pe_{i(k+1)}\right)\right) \cup Se_i\right],$$

(Some of the terms may be empty).

Lemma 2.5. *Let $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ be a finite P_0P-lattice of order n whose center B contains m atoms, and let h be an automorphism of P. Then* *(a)* $h(Fe_i) = Fh(e_i)$, $i = 1\cdots, n-1$, *where*

$$\begin{aligned} F(e_i) &= \{a : a \text{ atom of } B \text{ and } ae_i = 0\} \\ h(Fe_i) &= \{a : a \text{ atom of } B \text{ and } ah(e_i) = 0\} \end{aligned}$$

(b) $h\left(P^{s(j)}_{e_{ik}}\right) = P^{s(j)}_{h(e_i)k}$, $i = 1\cdots, n-2$; $k = i+1, \cdots, n-2$ *and* $j = 1\cdots, 2^{k-i}$

Proof. It is easy to see that h permutes the atoms of B. To prove (a) note that $ae_i = 0$ iff $h(a)h(e_i) = 0$. Hence $a \in Fe_i$ iff $h(a) \in Fh(e_i)$, i.e. (a) holds.

To prove (b), let $a \in P^{s(j)}_{e_{ik}}$, then $a \in Pe_{ik}$ and $ae_{l-1} = ae_l$ iff $l \in s(j)$, so

$$h(a)h(e_i) < h(a),\ h(a)h(e_k) < h(a) \text{ and } h(a)h(e_{l-1}) = h(a)h(e_l) \text{ iff } l \in s(J).$$

Hence $h(a) \in P^{s(j)}h(e_i)k$. The converse implication is similar. □

Corollary 1. *If $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ is a P_0P-lattice of order n and if h is an automophrism of P with $h(e_i) = e_i$, $i = 1\cdots, n-1$, then h permutes each of the sets Fe_i. That is, $h(Fe_i) = Fe_i$, $i = 1\cdots, n-2$ and $P^{s(j)}_{e_{ik}}$, i.e. $h\left(P^{s(j)}_{e_{ik}}\right) = P^{s(j)}_{e_{ik}}$, $i = 1\cdots, n-2$, $k = i\cdots, n-1$ and $j = 1\cdots, 2^{k-i}$.*

Remark 2.2. It is clear using Remarks 2.1 and Corollary 2.1 that if h is an automorphism of P with $h(e_i) = e_i$, $i = 1, \cdots, n-1$, then h permutes each of the following sets

(f'_1) $Pe_{ik}, k = i+1, \cdots, n-1$

(f'_2) $P^{s(j)}_{e_{ik}} - P_{i(k+1)}, k = i+1, \cdots, n-1, j = 1, \cdots, 2^{k-i}$

(f'_3) $Se_i = Fe_{i-1} - (Fe_i \cup Pe_i), i = 1, \cdots, n-1.$

Lemma 2.6. *Let $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ be a P_0P-lattice with a finite center B. Let A be the set of all atoms in B. Then*

(i) $e_i! = \vee(A - Fe_i)$, $i = 1, \cdots, n-1$

$$(ii)\quad e_k \Rightarrow e_j = \vee\left(Fe_k \cup De_j \cup \left(\bigcup_{i=1}^{j} \cup \{P^{s(l)}e_{ik} : s(l) \supseteq \{j+1, \cdots, k\}\}\right)\right),$$

$k > j$, *where* $De_j = \{a : a \in A,\ ae_j = a\}$ *and* Fe_i *and* $P^{s(j)}_{e_{ik}}$ *are defined in Definition 2.2.*

Proof.

(i) Let $a \in Fe_i$, then $ae_i = 0$, hence $a \le e_i \Rightarrow 0$ which implies $\bigvee Fe_i \le e_i \Rightarrow 0$.

Now, $\bigvee(A - Fe_i) = \overline{(\bigvee Fe_i)} \geq \overline{(e_i \Rightarrow 0)} = e_i!$, i.e. $\bigvee(A - Fe_i) \geq e_i!$.On the order hand let $a \leq \bigvee(A - Fe_i)$, $a \in A$, then $0 < ae_i \leq a$. Hence, $a(e_i!) \leq a$, which implies $a(e_i!) = a$ (a is an atom). This implies $a \leq e_i!$. Thus $\bigvee(A - Fe_i) \leq e_i!$.

(ii) Let $a \leq \vee\left(Fe_k \cup De_j \cup \left(\bigcup_{i=1}^{j} \cup \{P^{s(l)}e_{ik} : s(l) \supseteq \{j+1, \cdots, k\}\}\right)\right)$, then either $a \in Fe_k$, hence $ae_k = 0 < e_j$ which implies $a \leq e_k \Rightarrow e_j$, or $a \in De_j \subseteq De_k$, hence $a = ae_k = ae_j < e_j$, which implies $a \leq e_k \Rightarrow e_j$, or $a \in P^{s(j)}e_{i_0k}$, where $s(l) \supseteq \{j+1, \cdots, k\}$ and $1 \leq i_0 \leq j$. Hence, $0 < ae_k = ae_{k-1} = \cdots = ae_{j+1} = ae_j < e_j$. i.e. $ae_k < e_j$, which implies $a \leq e_k \Rightarrow e_j$, hence, $\vee\left(Fe_k \cup De_j \cup \left(\bigcup_{i=1}^{j} \cup \{P^{s(l)}e_{i_0k} : s(l) \supseteq \{j+1, \cdots, k\}\}\right)\right) \leq e_k \Rightarrow e_j$. Suppose equality does not hold, then there exists at least one $a \in A$, $a \leq e_k \Rightarrow e_j$ and $a \notin \left(Fe_k \cup De_j \cup \left(\bigcup_{i=1}^{j} \cup \{P^{s(l)}e_{ik} : s(l) \supseteq \{j+1, \cdots, k\}\}\right)\right)$, $k > j$. Hence, $ae_k \neq 0$, $ae_j \neq 0$ and $ae_k \neq ae_j$ ($ae_j \neq 0$ because $j > i$) and this implies $ae_k \nleq e_j$ (because $ae_k < e_j$ implies $ae_k = ae_j$, which contradicts $ae_k \neq ae_j$) Since $ae_k \nleq e_j$, $a \nleq e_k \Rightarrow e_j$. Contradiction. Hence,

$e_k \Rightarrow e_{j=} \vee\left(Fe_k \cup De_j \cup \left(\bigcup_{i=1}^{j} \cup \{P^{s(l)}e_{ik} : s(l) \supseteq \{j+1, \cdots, k\}\}\right)\right)$,
$k > j$ □

Lemma 2.7. *Let $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ be a P_0P-lattice with atomic center $B \approx 2^m$, then for any $x \in P$ and for $k < j$, $D_i(x)(e_k \Rightarrow e_j) \leq D_k(x)$.*

Proof. Let $a \leq D_i(x)(e_k \Rightarrow e_j)$, $k > j$, then $a \leq D_j(x) = x!(e_j \Rightarrow x)$ and $a \in \left(Fe_k \cup De_j \cup \left(\bigcup_{i=1}^{j} \bigcup \left\{P_{e_{ik}}^{s(l)} : s(l) \supseteq \{j+1, \cdots, k\}\right\}\right)\right)$

(i) If $a \in Fe_k$, then $ae_k = 0$ which implies $a \leq (e_k \Rightarrow 0) \leq e_k \Rightarrow x$ ($a \leq D_j(x)$ implies $a \leq x!$ and $a \leq e_j \Rightarrow x$).

(ii) if $a \in De_j$, then $ae_k = ae_j < x$ (because $a \leq e_j \Rightarrow x$). Hence $a \leq (e_k \Rightarrow x)$ which implies $a \leq D_k(x)$.

(iii) if $a \in \left\{P_{e_{ik}}^{s(j)} : s(l) \supseteq \{j+1, \cdots, k\}\right\}$ then $a = a_k = ae_{k-1} = \cdots = ae_{j+1} = ae_j \leq x$. Hence $a \leq e_k \Rightarrow x$, thus $a \leq x!\,(e_k \Rightarrow x) = D_k(x)$. □

Lemma 2.8. *if $h_0 : B \to B$ is an automorphism of a finite Boolean algebra B, and $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ is a P_0-lattice having another chain base*

$$0 = e_0' < e_1' < \cdots < e_{n-1}' = 1,$$

and if h_0 satisfies

$$\text{(2.1)} \qquad \begin{cases} h_0 (Fe_i) = Fe_i', \\ h_0 \left(P_{e_{ik}}^{s(j)}\right) = P_{e_{ik}'}^{s(j)} \end{cases}$$

$i = 1, \cdots, n-2$, $k = i+1, \cdots, n-1$ and $j = 1, \cdots, 2^{k-i}$, then

(a) $h_0 (e_k \Rightarrow e_j) = \left(e_k' \Rightarrow e_j'\right)$,
(b) $h_0 (e_j!) = e_j!$;
(c) $\bigvee\limits_{j=k}^{n-1} h_0 (D_j (x))\, e_j' = h_0 (D_k (x))$,
(d) $h_0 (D_j (x)) \left(e_k' \Rightarrow e_j'\right) \leq h_0 (D_k (x))$, *for $k > j$.(by Lemma 2.6)*

Proof.
(a) It is clear for $k \leq j$ $(e_k \Rightarrow e_j = 1)$. Let $k > j$. Then (by Lemma 2.6)

$$h_0 (e_k \Rightarrow e_j) =$$

$$h_0 \left(\vee \left(Fe_k \cup De_j \cup \left(\bigcup_{i=1}^{j} \bigcup \left\{ P_{e_{ik}}^{s(l)} : s(l) \supseteq \{j+1, \cdots, k\} \right\} \right) \right) \right)$$

$$= \vee \left(h_0 (Fe_k) \cup h ({}_0De_j) \cup \left(\bigcup_{i=1}^{j} \bigcup \left\{ h_0 \left(P_{e_{ik}}^{s(l)}\right) : s(l) \supseteq \{j+1, \cdots, k\} \right\} \right) \right)$$

$$= \vee \left(Fe_k' \cup De_j' \cup \left(\bigcup_{i=1}^{j} \bigcup \left\{ P^{s(l)} e_{ik} : s(l) \supseteq \{j+1, \cdots, k\} \right\} \right) \right)$$

$$= e_k' \Rightarrow e_j'.$$

(b) By Lemma 2.11,

$$h_0 (e_i!) = h_0 (\vee (A - Fe_i)) = \vee (A - h_0 (Fe_i)) = \vee (A - Fe_i) = e_i'!.$$

(c) For $k \leq j$, $h_0 (D_j (x))\, e_j'! \leq h_0 (D_j (x)) \leq h_0 (D_k (x))$ (by property 4 (a) in 1). Hence $\bigvee\limits_{j=k}^{n-1} h_0 (D_j (x))\, e_j'! \leq h_0 (D_j (x))$. Let $h_0 (a) \leq h_0 (D_k (x))$ where a is an atom of B. Then either

(i) $0 \neq h_0(a)\, e'_k \leq h_0(a)$, or (ii) $h_0(a)\, e'_k = 0$.

If (i) holds, then $h_0(a) \leq e'_k!$. Hence $h_0(a) \leq h_0(D_k(x))\, e'_k \leq \bigvee_{j=k}^{n-1} h_0(D_j(x))\, e'_j!$ and (c) holds.

Suppose (ii) holds. Now, $h_0(a) \leq h_0(D_k(x)) = h_0(x!\,(e_k \Rightarrow x))$ implies $h_0(a) \leq h_0(x!)$ which implies $a \leq x!$ (h_0 is an automorphism). $a \leq x!$ implies either: $(l_1)\, a \leq x$, or $(l_2)\, ax \neq 0$, or $(l_3)\, x < a$. If (l_1) is satisfied, then $a \leq !x = D_{n-1}(x)$ and $h_0(a) \leq h_0(D_{n-1}(x)) \leq \bigvee_{j=k}^{n-1} h_0(D_j(x))\, e'_j!$ and (c) holds. If (l_2) is satisfied, then $0 \neq ax = \bigvee_{j=1}^{n-1} (D_j(x))\, e_j = \bigvee_{j=1}^{n-1} aD_j(x)\, e_j = \bigvee_{j>k}^{n-1} a(D_j(x)\, e_j)$ (because $h_0(a)\, e'_k = 0$ which implies $ae_k = 0$). Hence, there exists at least one index $j_0 > k$ such that $aD_{j_0}(x)\, e_j \neq 0$, and this implies $aD_{j_0}(x) \neq 0$ and $ae_j \neq 0$. Since $aD_{j_0}(x)$ is a Boolean element and a is an atom, $a \leq D_{j_0}(x)$. Now, $ae_{j_0} \neq 0$ implies $ae_{j_0} \leq a$ and $a \leq e_j!$. Hence, $h_0(a) \leq h_0(D_{j_0}(x))$ and $h_0(a) \leq h_0(e_{j_0}!) = e'_{j_0}!$ (by part (b)) which implies $h_0(a) \leq h_0(D_{j_0}(x))\, e'_{j_0}! \leq \bigvee_{j=k}^{n-1} h_0(D_j(x))\, e'_j!$, and (c) again holds.

if (l_3) is satisfied, then $x < a$ implies $ax = x$ and the proof is similar to the one above.

(d) The proof follows by Lemma 2.7 and part (a). □

Theorem 2.1. *If $h_0 : B \to B$ is an automorphism of a finite Boolean algebra B and $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ is a P_0P-lattice having another chain base*

$$0 = e'_0 < e'_1 < \cdots < e'_{n-1} = 1,$$

then there exists an automorphism h of P such that $h|_B = h_0$ and $h(e_i) = e'_i$, $i = 1, \cdots, n-1$ if and only if h_0 satisfies (2.1) of Lemma 2.8, i.e. h_0 satisfies

$$h_0(Fe_i) = Fe'_i,\ i = 1, \cdots, n-1 \text{ and } h_0\left(P^{s(j)}_{e_{ik}}\right) = P^{s(j)}_{e_{ik}},\ i = 1, \cdots, n-2,$$

$k = i+1, \cdots, n-1$ and $j = 1, \cdots, 2^{k-i}$.

Proof. Suppose h_0 satisfies (2.1) and let $x, y \in P$, then x and y have a monotonic representation, $x = \bigvee_{i=1}^{n-1} D_i(x)\, e_i$ and $y = \bigvee_{i=1}^{n-1} D_i(y)\, e_i$, where $D_i(x) = x!\,(e_i \Rightarrow x)$ and $D_i(y) = y!\,(e_i \Rightarrow y)$. Let $h : P \to P$ be defined by $h(x) = \bigvee_{i=1}^{n-1} h_0(D_i(x))\, e'_i$.

(1) First we shall prove that $h|_B = h_0$. For $b \in B$, $b = \bigvee_{i=1}^{n-1} D_i(b)\, e_i = \bigvee_{i=1}^{n-1} be_i$ (by 4(d)). Hence, $h(b) = \bigvee_{i=1}^{n-1} h_0(b)\, e_i' = h_0(b) \bigvee_{i=1}^{n-1} e_i' = h_0(b)\, 1 = h_0(b)$, which implies $h|_B = h_0$.

(2) Now we shall prove that $D_k'(h(x)) = h_0(D_k(x))$.

$$\begin{aligned}
D_k'(h(x)) &= D_k'\left(\bigvee_{j=1}^{n-1} h_0(D_j(x))\, e_j'\right) = \bigvee_{j=1}^{n-1} D_k'(h_0(D_j(x)))\, D_k'(e_j') \\
&= \bigvee_{j=1}^{n-1} h_0(D_j(x))\, D_k'(e_j') = \bigvee_{i=1}^{n-1} h_0(D_i(x))\, e_i' \\
&= \bigvee_{j=1}^{n-1} h_0(D_j(x))\, e_j'!\,(e_k' \Rightarrow e_j) \quad \text{(by 4 (d) and 4 (e))} \\
&= \bigvee_{j=1}^{k-1} h_0(D_j(x))\, e_j'!\,(e_k' \Rightarrow e_j') \vee \bigvee_{j=k}^{n-1} h_0(D_j(x))\, e_j'!.
\end{aligned}$$

Now, by part (d) of Lemma 2.8
$h_0(D_j(x))\left(e_k' \Rightarrow e_j'\right) e_j' \leq h_0(D_k(x))$ for $k > j$. Hence,

$$\bigvee_{j=1}^{n-1} h_0(D_j(x))\, e_i'!\,(e_k' \Rightarrow e_j') \leq h_0(D_k(x)).$$

$\bigvee_{j=1}^{n-1} h_0(D_j(x))\, e_i'! = h_0(D_k(x))$ by (4 (c) and 4 (e)). Hence, $D_k'(h(x)) = \bigvee_{j=1}^{k-1} h_0(D_j(x))\, e_i'!\left(e_k' \Rightarrow e_j'\right) \vee \bigvee_{j=k}^{n-1} h_0(D_j(x))\, e_j'! = h_0(D_k(x))$.

(3) We shall prove that $h(e_j) = e_j'$ for $j = 1, \cdots, n-1$.

$$\begin{aligned}
h(e_j) &= \bigvee_{i=1}^{n-1} h_0(D_i(e_j))\, e_i' = \bigvee_{i=1}^{n-1} h_0(e_j!\,(e_i \Rightarrow e_j))\, e_i' \qquad (by\ 4.\ \text{(e)}) \\
&= \bigvee_{i=1}^{n-1} h_0(e_j!)\, h_0(e_i \Rightarrow e_j)\, e_i' = h(e_j) \\
&= \bigvee_{i=1}^{n-1} e_j'!\,(e_i' \Rightarrow e_j')\, e_i' \quad \text{(by parts (b) and (c) of Lemma 2.8)} \\
&= \bigvee_{i=1}^{n-1} (e_j')\, e_i' \vee \bigvee_{i=j+1}^{n-1} (e_j!)\,(e_i' \Rightarrow e_j')\, e_i'
\end{aligned}$$

$$= \bigvee_{i=1}^{j} e'_i \vee \bigvee_{i=j+1}^{n-1} e_j!\left(e'_i \Rightarrow e'_j\right) e'_i, \left(e'_i \leq e'_j \leq e'_j! \text{ for } i \leq j\right)$$

$$= e'_j \vee \bigvee_{i=j+1}^{n-1} (e'_j)!\left(e'_i \Rightarrow e'_j\right) e'_i = e'_j.$$

(4) We prove h is a lattice homomrphism, i.e. $h(x \vee y) = h(x) \vee h(y)$ and $h(xy) = h(x)\, h(y)$ for all $x, y \in P$.
This easy proof is omitted.
(5) To prove h is one-to-one, Let $x, y \in P$, $x = \bigvee_{i=1}^{n-1} D_i(x)\, e_i$, $y = \bigvee_{i=1}^{n-1} D_i(y)\, e_i$ and $h(x) = h(y)$. Hence, $\bigvee_{i=1}^{n-1} D'_i(h(x))\, e'_i = h(x) = h(y) = \bigvee_{i=1}^{n-1} D_I(h(y))\, e'_i$. Since the representation is unique, $D'_i(h(x)) = D'_i(h(y))$, $i = 1, \cdots n-1$. Hence, (2) implies that $h_0(D_i(x)) = D'_i(h(x)) = D'_i(h(y)) = h_0(D_i(y))$, $i = 1, \cdots, n-1$ and consequently $D_i(x) = D_i(y)$, $i = 1, \cdots, n-1$. Hence, $x = y$.
(6) To prove h is onto, let $y = \bigvee_{i=1}^{n-1} D'_i(y)\, e'_i$, be monotonic representation of $y \in P$. Let $x = \bigvee_{i=1}^{n-1} h_0^{-1}(D'_i(y))\, e_i$. Since h_0 is an automorphism, $x = \bigvee_{i=1}^{n-1} h_0^{-1}(D'_i(y))\, e_i$ is a monotonic representations. of x. Hence $h(x) = \bigvee_{i=1}^{n-1} h_0(D_i(x))\, e'_i = \bigvee_{i=1}^{n-1} h_0\left(h_0^{-1}(D'_i(y))\right) e'_i = \bigvee_{i=1}^{n-1} D'_i(y)\, e'_i = y$.

(7) We shall prove that $h(x \Rightarrow y) = h(x) \Rightarrow h(y)$ for all $x, y \in P$.
Let $b = h(x) \Rightarrow h(y)$, hence $bh(x) \leq h(y)$ which implies $h_0^{-1}(b)\, x \leq y$ and this implies $h_0^{-1}(b) \leq x \Rightarrow y$. Hence, $b \leq h_0(x \Rightarrow y) = h(x \Rightarrow y)$ (by (1)). Hence, $h(x) \Rightarrow h(y) \leq h(x \Rightarrow y)$. Now, $h(x)\, h(x \Rightarrow y) = h(x(x \Rightarrow y)) \leq h(y)$. Hence, $h(x \Rightarrow y) \leq h(x) \Rightarrow h(y)$ and (7) is proved. Thus we have proved that h is P_0P-lattice automorphism (Definition 2.1).
The converse follows by Lemma 2.8. □

Remark 2.3. Let A be the set of all atoms in a finite *P_0P-lattice* $P = \langle B, e_0, \cdots, e_{n-1} \rangle$ with $|A|$ (the order of A) $= m$. Let $s_i = |Se_i|$, $p_{ikj} = \left|P_{e_{ik}}^{s(j)} - Pe_{i(k+1)}\right|$, $i = 2, \cdots, n-1$; $k = i, \cdots, n-2$; $j = 1, \cdots, 2^{k-i}$. Then using Remark 2.1 (vi), $m = \sum_{i=1}^{n-2}\left(s_i + \sum_{k=i}^{n-2}\left(\sum_{j=1}^{2^{k-i}} p_{ikj}\right)\right)$. Some of the numbers s_i, p_{ikj} may be zero.

Remark 2.4. It is well-known that if A is a set of m elements (m is finite), then the number of permutations on A that leave a set $S \subset A$ of r elements ($r < m$) fixed is $r!\,(m-r)!$.

Theorem 2.2. *Let $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ be a P_0P-lattices of order n, with the center $B \approx 2^m$, where m is finite. Then the number of automorphisms of P that leave the chain $e_0, \cdots, e_{n-1}$ fixed is $\prod_{i=1}^{n-2}\prod_{k=i}^{n-2}\prod_{j=1}^{2^{k-i}} (p_{ikj})!\,(s_i)!$, where $s_i = |Se_i|$, and $p_{ikj} = \left|P_{e_{ik}}^{s(j)} - Pe_{i(k+1)}\right|$, $i = 1\cdots, n-2$; $k = i\cdots, n-2$; $j = 1\cdots, 2^{k-i}$.*

Proof. Let A be the set of all atoms of B. Hence $|A| = m$. By Remark 2.3, $m = \sum_{i=1}^{n-2}\left(s_i + \sum_{k=i}^{n-2}\left(\sum_{j=1}^{2^{k-i}} p_{ikj}\right)\right)$, where $s_i = |Se_i|$, and $p_{ikj} = \left|P_{s_{ik}}^{s(j)} - Pe_{i(k+1)}\right|$ $i = 1\cdots, n-2$; $k = i\cdots, n-2$; $j = 1\cdots, 2^{k-i}$. Hence, by Theorem 2.1 and Corollary 2.1, the only automorphisms of B that can be extended to automorphisms of P leaving the chain $e_0, \cdots, e_{n-1}$ fixed are the automorphisms that leave fixed each of the sets Se_i and $P_{e_{ik}}^{s(j)} - Pe_{i(k+1)}$ and $i = 1\cdots, n-2$; $k = i\cdots, n-2$; $j = 1\cdots, 2^{k-i}$. Hence, the number of automorphisms of P leaving the chain $e_0, \cdots, e_{n-1}$ fixed is greater than or equal to: $\prod_{i=1}^{n-2}\prod_{k=i}^{n-2}\prod_{j=1}^{2^{k-i}} (p_{ikj})!\,(s_i)!$. Since the extension of h_0 is unique, Theorem 2.1 implies that the number is exactly equal to $\prod_{i=1}^{n-2}\prod_{k=i}^{n-2}\prod_{j=1}^{2^{k-i}} (p_{ikj})!\,(s_i)!$. □

Corollary 2. *Let $P = \langle B, e_0, \cdots, e_{n-1}\rangle$ be a P_0P-lattices of order n, with the center $B \approx 2^m$, where m is finite. Let the chain $e_0, \cdots, e_{n-1}$ satisfy the condition:*
(if a is an atom of B and $0 \neq ae_i$, then $0 \neq ae_j$ for all $1 \leq j \leq i \leq n-2$).

Then the number of automorphisms of P that leave the chain $e_0, \cdots, e_{n-1}$ fixed is $\prod_{k=1}^{n-2}\prod_{j=1}^{2^{k-1}} (p_{kj})!\,(s_1)!\,(d_1)!$, $k = 1\cdots, n-2$; $j = 1\cdots, 2^{k-1}$ where $s_1 = |Se_1|$, and $d_1 = |Fe_1|$ and $p_{kj} = \left|P_{e_k}^{s(j)} - Pe_{(k+1)}\right|$.

Proof. It is clear that if the chain $e_0, \cdots, e_{n-1}$ satisfies the above condition, then $Fe_1 = Fe_2 = \cdots = Fe_n$, $Se_2 = \cdots = Se_{n-2} = \phi$, and i takes only the value 1. Hence, using Remark 2.1 (VI), A has the partition

$$A = Se_1 \cup Fe_1 \cup \bigcup_{k=1}^{n-2} \left(\bigcup_{j=1}^{2^{k-1}} \left(P_{e_k}^{s(j)} - Pe_{(k+1)} \right) \right),$$

where some of the terms may be empty. Hence, $m = s_1 + d_1 + \sum_{k=1}^{n-2} \left(\sum_{j=1}^{2^{k-1}} (p_{kj}) \right)$, where $s_1 = |Se_1|$, and $d_1 = |Fe_1|$ and $p_{kj} = \left| P_{e_k}^{s(j)} - Pe_{(k+1)} \right|$, $k = 1 \cdots, n-2$; $j = 1 \cdots, 2^{k-1}$. Hence, by theorem 2.2 the number of automorphisms of P that leave the chain $e_0, \cdots, e_{n-1}$ fixed is $\prod_{k=1}^{n-2} \prod_{j=1}^{2^{k-1}} (p_{kj})!\,(s_1)!\,(d_1)!$.

It is clear that Corollary 2.2 is exactly Theorem 2.2 of [5]. □

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