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GENERALIZED SOBOLEV SPACES AND PSEUDO-DIFFERENTIAL OPERATORS OF EXPONENTIAL TYPE ON THE LAGUERRE HYPERGROUP

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ABSTRACT. In this paper we introduce a generalized Sobolev spaces of exponential type $\mathbb{H}^s_{\exp}(\mathbb{K})$ on the Laguerre hypergroup and we investigate some properties of these spaces. We define a pseudo-differential operators of exponential type associated with a class of symbols operating naturally on the spaces $\mathbb{H}^s_{\exp}(\mathbb{K})$ and we prove a boundedness theorem of these operators.

1. INTRODUCTION

The Sobolev spaces were the object of several works and serve as a very useful tool in the theory of partial differential equations. The study of these spaces know a long history and has been exploited by many authors in different scopes from R. Adams [1], H. Bresis [8], M. T. Lacroix-Sentier [16] and others in the classical case $\mathbb{R}^n$ to H. Bahouri, P. Gérard and C. J. Xu on the Heisenberg group $\mathbb{H}^n$ [6], M. Assal and M. M. Nessibi on the Bessel hypergroup and on the dual of the Laguerre hypergroup [3, 4] and N. Ben Salem and A. O. A. Salem with the Jacobi-Dunkl operator [5]. In the present work we introduce Sobolev spaces of exponential type $\mathbb{H}^s_{\exp}(\mathbb{K})$ generalizing the classical ones on the Laguerre hypergroup $\mathbb{K} = [0, +\infty[\times\mathbb{R}$ based on the generalized Schwartz space $S_{*,\exp}$ and we define pseudo-differential operators associated with a class of

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symbols of exponential type naturally acting on the so called Sobolev type spaces.

Throughout this paper we fix $\alpha \geq 0$ and we consider the following system of partial differential operators:

$$\begin{cases} D_1 &= \dfrac{\partial}{\partial t}, \\ D_{\mathbb{K}} &= \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2\dfrac{\partial^2}{\partial t^2}\ ; \qquad (x,t) \in]0,\infty[\times\mathbb{R}. \end{cases}$$

For $\alpha = n-1$; $n \in \mathbb{N}\backslash\{0\}$, the operator $D_{\mathbb{K}}$ is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$. We denote by $\varphi_{\lambda,m}$, $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$, the unique solution of the following system:

$$\begin{cases} D_1 u(x,t) &= i\lambda u(x,t), \\ D_{\mathbb{K}} u(x,t) &= -4|\lambda|(m+\frac{\alpha+1}{2})u(x,t); \\ u(0,0) &= 1, \\ \dfrac{\partial u}{\partial x}(0,t) &= 0, \qquad \textit{for all } t\in\mathbb{R}. \end{cases}$$

One knows that $\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^\alpha(|\lambda|x^2)$, where $\mathcal{L}_m^\alpha$ is the Laguerre function defined on $\mathbb{R}_+$ by $\mathcal{L}_m^\alpha(x) = e^{-\frac{x}{2}}\dfrac{L_m^\alpha(x)}{L_m^\alpha(0)}$ and L_m^α is the Laguerre polynomial of degree m and order α ([15], [9], [11], [13]).

We recall that for $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$ and for a suitable function $f : \mathbb{K} \longrightarrow \mathbb{C}$ the Fourier-Laguerre transform $\mathcal{F}(f)(\lambda,m)$ of f at (λ,m) is defined by ([17], [19, 20], [10]):

$$\mathcal{F}(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)f(x,t)d\mu_\alpha(x,t) \tag{1.1}$$

where $d\mu_\alpha(x,t) = \dfrac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}$.

It has been proved in [17, Theorem II.1] that the Fourier-Laguerre transform is a topological isomorphism from $S_*(\mathbb{K})$ onto $S(\mathbb{R}\times\mathbb{N})$ where

- $S_*(\mathbb{K})$ is the Schwartz space of functions $\psi : \mathbb{R}^2 \longrightarrow \mathbb{C}$ even with respect to the first variable, $\mathcal{C}^\infty$ on $\mathbb{R}^2$ and rapidly decreasing together with all their derivatives; i.e. for all $k,p,q\in\mathbb{N}$ we have

(1.2) $$\mathcal{V}_{k,p,q}(\psi)=\sup_{(x,t)\in\mathbb{K}}\Big\{(1+x^2+t^2)^k\Big|\frac{\partial^{p+q}}{\partial x^p\partial t^q}\psi(x,t)\Big|\Big\}<\infty.$$

- $\mathcal{S}(\mathbb{R}\times\mathbb{N})$ the space of functions $\Psi:\mathbb{R}\times\mathbb{N}\longrightarrow\mathbb{C}$ satisfying :

  i) For all $m,p,q,r,s\in\mathbb{N}$, the function

$$\lambda\longmapsto\lambda^p\Big(|\lambda|(m+\frac{\alpha+1}{2})\Big)^q\Lambda_1^r\Big(\Lambda_2+\frac{\partial}{\partial\lambda}\Big)^s\Psi(\lambda,m)$$

is bounded and continuous on $\mathbb{R}$, $\mathcal{C}^\infty$ on $\mathbb{R}^*=\mathbb{R}\backslash\{0\}$ and such that the left and the right derivatives at zero exist.

  ii) For all $k,p,q\in\mathbb{N}$, we have

(1.3)
$$\widetilde{\mathcal{V}}_{k,p,q}(\Psi)=\sup_{(\lambda,m)\in\mathbb{R}^*\times\mathbb{N}}\Big\{\big(1+\lambda^2(1+m^2)\big)^k\Big|\Lambda_1^p\Big(\Lambda_2+\frac{\partial}{\partial\lambda}\Big)^q\Psi(\lambda,m)\Big|\Big\}<\infty.$$

where

- $\Lambda_1\Psi(\lambda,m)=\frac{1}{|\lambda|}\Big(m\Delta_+\Delta_-\Psi(\lambda,m)+(\alpha+1)\Delta_+\Psi(\lambda,m)\Big)$.
- $\Lambda_2\Psi(\lambda,m)=\frac{-1}{2\lambda}\Big((\alpha+m+1)\Delta_+\Psi(\lambda,m)+m\Delta_-\Psi(\lambda,m)\Big)$.
- $\Delta_+\Psi(\lambda,m)=\Psi(\lambda,m+1)-\Psi(\lambda,m)$.
- $\Delta_-\Psi(\lambda,m)=\Psi(\lambda,m)-\Psi(\lambda,m-1)$, *if* $m\geq 1$ *and* $\Delta_-\Psi(\lambda,0)=\Psi(\lambda,0)$.

We note that $S_*(\mathbb{K})$ (resp. $S(\mathbb{R}\times\mathbb{N})$) equipped with the seminorms $\mathcal{V}_{k,p,q}$ (resp. $\widetilde{\mathcal{V}}_{k,p,q}$), $k,p,q\in\mathbb{N}$, is a Fréchet space ([17]).

It has been proved in [18] that $\mathbb{K}$ is induced with a convolution product $\#$ turning $(\mathbb{K},\#,i)$ into a commutative hypergroup in the sense of Jewett ([12], [7]) where i denotes the involution defined on $\mathbb{K}$ and given by $i(x,t)=(x,-t)$. Throughout this paper we denote by

- $N(x,t)=(x^2+|t|)^{1/2}$ the norm of $(x,t)\in\mathbb{K}$ and $\widetilde{N}(\lambda,m)=|\lambda|(m+\frac{\alpha+1}{2})$; $(\lambda,m)\in\widehat{\mathbb{K}}=\mathbb{R}\times\mathbb{N}$.
- $\mathcal{C}_*^\infty(\mathbb{K})$ the space of functions $f:\mathbb{R}^2\longrightarrow\mathbb{C}$, even with respect to the first variable and $\mathcal{C}^\infty$ on $\mathbb{R}^2$.

- $L^p(\widehat{\mathbb{K}}) = L^p(\widehat{\mathbb{K}}, d\gamma_\alpha)$, $1 \le p \le \infty$, the space of measurable functions g on $\widehat{\mathbb{K}}$ such that $\|g\|_{L^p(\widehat{\mathbb{K}})} < \infty$, where

$$\begin{aligned} \|g\|_{L^p(\widehat{\mathbb{K}})} &= \left(\int_{\widehat{\mathbb{K}}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{p}}, \qquad if \quad p \in [1,\infty[, \\ \|g\|_{L^\infty(\widehat{\mathbb{K}})} &= \operatorname*{ess\,sup}_{(\lambda,m)\in\widehat{\mathbb{K}}} |g((\lambda,m))|. \end{aligned}$$

$d\gamma_\alpha$ being the spectral measure on $\widehat{\mathbb{K}}$ defined by

$$d\gamma_\alpha(\lambda,m) = |\lambda|^{\alpha+1} L_m^\alpha(0) d\lambda \otimes \delta_m.$$

Definition 1.1.

- *The generalized translation operators* $\mathcal{T}^{(\alpha)}_{(\lambda,m)}$ *on* $\widehat{\mathbb{K}}$ *are given for a suitable function* f *by:*

$$\mathcal{T}^{(\alpha)}_{(\lambda,m)} f(\mu,n) = \sum_{j\in\mathbb{N}_{m,n}} f(\lambda+\mu, j) C_j^\alpha((\lambda,m)(\mu,n))$$

where

$$C_j^\alpha((\lambda,m)(\mu,n)) = \frac{L_j^\alpha(0)}{\Gamma(\alpha+1)} \int_0^\infty \mathcal{L}_m^\alpha\left(\left|\frac{\lambda}{\lambda+\mu}\right|x\right)\mathcal{L}_n^\alpha\left(\left|\frac{\mu}{\lambda+\mu}\right|x\right)\mathcal{L}_j^\alpha(x) x^\alpha dx$$

and

$$\mathbb{N}_{m,n} = \begin{cases} \{0,1,...,m+n\}, & if \quad \lambda+\mu \neq 0 \text{ and } \lambda\mu > 0, \\ \mathbb{N}, & if \quad \lambda+\mu \neq 0 \text{ and } \lambda\mu \le 0. \end{cases}$$

- *The generalized convolution product on* $\widehat{\mathbb{K}}$ *is defined for a suitable pair of functions* f *and* g *by:*

$$f \otimes g(\lambda,m) = \int_{\widehat{\mathbb{K}}} \mathcal{T}^{(\alpha)}_{(\lambda,m)} f(\mu,n) g(-\mu,n) d\gamma_\alpha(\mu,n) \qquad \text{for all } (\lambda,m) \in \widehat{\mathbb{K}}.$$

Proposition 1.1. *The following properties hold*

1) *For all* $(\lambda,m),(\mu,n) \in \widehat{\mathbb{K}}$ *and for an appropriate function* f *on* $\widehat{\mathbb{K}}$*, we have (see* [14]*)*
   (i) $\mathcal{T}^{(\alpha)}_{(0,0)} f(\lambda,m) = f(\lambda,m), \qquad \forall(\lambda,m) \in \widehat{\mathbb{K}}.$
   (ii) $\mathcal{T}^{(\alpha)}_{(\lambda,m)} f(\mu,n) = \mathcal{T}^{(\alpha)}_{(\mu,n)} f(\lambda,m), \qquad \forall(\lambda,m),(\mu,n) \in \widehat{\mathbb{K}}.$
   (iii) $[\mathcal{T}^{(\alpha)}_{(\lambda,m)} \varphi_{(.,.)}(x,t)](\mu,n) = \varphi_\lambda(x,t)\varphi_{(\mu,n)}(x,t)$, *for all* $(x,t) \in \mathbb{K}$ *and* $(\lambda,m),(\mu,n) \in \widehat{\mathbb{K}}$.

2) *Let* $f \in L^p(\widehat{\mathbb{K}})$, $1 \leq p \leq \infty$. *Then, for all* $(\lambda, m) \in \widehat{\mathbb{K}}$, *the function* $\mathcal{T}^{(\alpha)}_{(\lambda,m)} f$ *belongs to* $L^p(\widehat{\mathbb{K}})$ *and we have*

$$\|\mathcal{T}^{(\alpha)}_{(\lambda,m)} f\|_{L^p(\widehat{\mathbb{K}})} \leq \|f\|_{L^p(\widehat{\mathbb{K}})}.$$

3) *For* f *in* $L^p(\widehat{\mathbb{K}})$ *and* $g \in L^q(\widehat{\mathbb{K}})$, $1 \leq p, q \leq \infty$, *the function* $f \otimes g$ *belongs to* $L^r(\widehat{\mathbb{K}})$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, *and we have*

$$\|f \otimes g\|_{L^r(\widehat{\mathbb{K}})} \leq \|f\|_{L^p(\widehat{\mathbb{K}})} \|g\|_{L^q(\widehat{\mathbb{K}})}.$$

We finish this introductory section by remaining the definition of the homogeneous Sobolev-Laguerre type space $\dot{\mathcal{H}}^s(\mathbb{K})$ introduced in [2].

Definition 1.2. *Let* $s \in \mathbb{R}$. *The homogeneous Sobolev-Laguerre type space* $\dot{\mathcal{H}}^s(\mathbb{K})$ *is the set of all tempered distributions* f *such that*

$$\|f\|_{\dot{\mathcal{H}}^s(\mathbb{K})} = \left(\int_{\widehat{\mathbb{K}}} [\widetilde{N}(\lambda, m)]^s |\mathcal{F}u(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \right)^{1/2} < +\infty.$$

Finally, we mention that, C will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.

2. Generalized Sobolev-Laguerre Spaces of Exponential Type

Definition 2.1. *We define the generalized Schwartz space denoting by* $S_{*,\exp} := S_{*,\exp}(\mathbb{K})$ *the space of all* C^∞ *function* ϕ *on* $\mathbb{K}$ *such that, for all* $p, q \in \mathbb{N}$ *and* $k \geq 0$,

$$P_{p,q,k}(\phi) = \sup_{(x,t)\in\mathbb{K}} \left\{ e^{kN(x,t)} |D_1^p D_{\mathbb{K}}^q \phi(x,t)| \right\} < \infty$$

and

$$\Pi_{p,q,k}(\phi) = \sup_{(\lambda,m)\in\widehat{\mathbb{K}}} \left\{ e^{k\widetilde{N}(\lambda,m)} \left| \Lambda_1^p \left(\Lambda_2 + \frac{\partial}{\partial \lambda} \right)^q (\mathcal{F}\phi)(\lambda, m) \right| \right\} < \infty.$$

We denote by $S'_{*,\exp} := S'_{*,\exp}(\mathbb{K})$ *the dual space of* $S_{*,\exp}$.

Note that for all $k \in \mathbb{N}$, $\phi(x,t) = e^{-\frac{k}{2}N(x,t)}$ belongs to $S_{*,\exp}$.

Definition 2.2. *Let $s \in \mathbb{R}$. We define the generalized Sobolev-Laguerre space of exponential type $\mathbb{H}^s_{\exp}(\mathbb{K})$ as the set of all generalized distribution $u \in S'_{*,\exp}$ such that*

$$\tilde{u}_s(\lambda, m) := \exp((s/2)\widetilde{N}(\lambda, m))\mathcal{F}u(\lambda, m) \in L^2(\widehat{\mathbb{K}}).$$

Remark 2.1. 1) *Equipped with the norm*

$$\|u\|_{\mathbb{H}^s_{\exp}(\mathbb{K})} := \|\tilde{u}_s\|_{L^2(\widehat{\mathbb{K}})} = \left(\int_{\widehat{\mathbb{K}}} \exp(s\widetilde{N}(\lambda, m))|\mathcal{F}u(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\right)^{1/2}$$

$\mathbb{H}^s_{\exp}(\mathbb{K})$ *is a Banach space.*

2) $\mathbb{H}^s_{\exp}(\mathbb{K})$ *is a Hilbert space with inner product*

$$(u, v)_{\mathbb{H}^s_{\exp}(\mathbb{K})} = \int_{\widehat{\mathbb{K}}} \exp(s\widetilde{N}(\lambda, m))\mathcal{F}u(\lambda, m)\overline{\mathcal{F}v(\lambda, m)}d\gamma_\alpha(\lambda, m).$$

3) a) $S_{*,\exp} \subset \mathbb{H}^s_{\exp}(\mathbb{K})$, *for all $s \in \mathbb{R}$.*

b) $\mathbb{H}^0_{\exp} = L^2(\mathbb{K})$.

c) $\mathbb{H}^{s_1}_{\exp}(\mathbb{K}) \subseteq \mathbb{H}^{s_2}_{\exp}(\mathbb{K})$, *for all $s_1 \geq s_2$.*

d) $\mathbb{H}^s_{\exp}(\mathbb{K}) \subseteq \dot{\mathcal{H}}^s(\mathbb{K})$, *for all $s \leq 0$.*

e) $\dot{\mathcal{H}}^s(\mathbb{K}) \subseteq \mathbb{H}^s_{\exp}(\mathbb{K})$, *for all $s \geq 0$.*

Proposition 2.1. *Let P a linear partial differential operator with constant coefficients. Then P maps continuously $\mathbb{H}^{s_1}_{\exp}(\mathbb{K})$ into $\mathbb{H}^{s_2}_{\exp}(\mathbb{K})$, for all $s_1 > s_2$.*

Proof. If P has the form $P = a_k D^k_{\mathbb{K}}$ and $u \in \mathbb{H}^{s_1}_{\exp}(\mathbb{K})$, then

$$|Pu\|^2_{\mathbb{H}^{s_2}_{\exp}(\mathbb{K})} = \int_{\widehat{\mathbb{K}}} \exp(s_2\widetilde{N}(\lambda, m))\Big|a_k\mathcal{F}(D^k_{\mathbb{K}}u)(\lambda, m)\Big|^2 d\gamma_\alpha(\lambda, m)$$

$$= C|a_k|^2 \int_{\widehat{\mathbb{K}}} \Big\{\widetilde{N}^{2k}(\lambda, m)\exp((s_2 - s_1)\widetilde{N}(\lambda, m))\Big\}\times$$

$$\Big\{\exp(s_1\widetilde{N}(\lambda, m))|\mathcal{F}u(\lambda, m)|^2\Big\}d\gamma_\alpha(\lambda, m)$$

$$\leq C|a_k|^2 \sup_{(\lambda,m)\in\widehat{\mathbb{K}}} \Big\{\widetilde{N}^{2k}(\lambda, m)\exp((s_2 - s_1)\widetilde{N}(\lambda, m))\Big\}\times$$

$$\int_{\widehat{\mathbb{K}}} \exp(s_1 \widetilde{N}(\lambda, m)) |\mathcal{F}u(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)$$

$$= C\|u\|^2_{\mathbb{H}^{s_1}_{\exp}(\mathbb{K})}.$$

The result remaind valid for $P = \sum_{k=1}^{n} a_k D^k_{\mathbb{K}}$; $n \in \mathbb{N}$. □

Example. Let $s \in \mathbb{R}$. The operator $\exp(\sqrt{1 - D_{\mathbb{K}}}) : \mathbb{H}^s_{\exp}(\mathbb{K}) \longrightarrow \mathbb{H}^{s-1}_{\exp}(\mathbb{K})$ defined by

$$\exp(\sqrt{1 - D_{\mathbb{K}}})u = \mathcal{F}^{-1}\Big(\exp(\sqrt{1 + \widetilde{N}^2(\lambda, m)})\mathcal{F}u\Big)$$

is an isomorphism and its inverse is the operator defined, for $v \in \mathbb{H}^{s-1}_{\exp}(\mathbb{K})$, by

$$\exp(-\sqrt{1 - D_{\mathbb{K}}})v = \mathcal{F}^{-1}\Big(\exp(-\sqrt{1 + \widetilde{N}^2(\lambda, m)})\mathcal{F}v\Big).$$

Proof. Let $u \in \mathbb{H}^s_{\exp}(\mathbb{K})$. Then we have

$$\|\exp(\sqrt{1 - D_{\mathbb{K}}})u\|^2_{\mathbb{H}^{s-1}_{\exp}(\mathbb{K})} =$$

$$\int_{\widehat{\mathbb{K}}} \exp((s-1)\widetilde{N}(\lambda, m)) \exp(\sqrt{1 + \widetilde{N}^2(\lambda, m)}) |\mathcal{F}u(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \leq$$

$$C\|u\|^2_{\mathbb{H}^s_{\exp}(\mathbb{K})}$$

where $C = \sup_{(\lambda, m) \in \widehat{\mathbb{K}}} \exp(\sqrt{1 + \widetilde{N}^2(\lambda, m)}) - \widetilde{N}(\lambda, m)) = e.$

Reciprocally, if $v \in \mathbb{H}^{s-1}_{\exp}(\mathbb{K})$, one has

$$\|\exp(-\sqrt{1 - D_{\mathbb{K}}})v\|^2_{\mathbb{H}^s_{\exp}(\mathbb{K})} =$$

$$\int_{\widehat{\mathbb{K}}} \exp(s\widetilde{N}(\lambda, m)) \exp(-\sqrt{1 + \widetilde{N}^2(\lambda, m)}) |\mathcal{F}u(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) =$$

$$\int_{\widehat{\mathbb{K}}} \exp(\widetilde{N}(\lambda, m) - \sqrt{1 + \widetilde{N}^2(\lambda, m)}\ ) \times \exp((s-1)\widetilde{N}(\lambda, m)) |\mathcal{F}u(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)$$

$\leq \sup_{(\lambda,m)\in\widehat{\mathbb{K}}} \exp(\widetilde{N}(\lambda,m) - \sqrt{1+\widetilde{N}^2(\lambda,m)})\|u\|^2_{\mathbb{H}^{s-1}_{\exp}(\mathbb{K})}$. This completes the proof. □

Note that, using a standard arguments one can prove easily that the topological dual $(\mathbb{H}^s_{\exp}(\mathbb{K}))'$ of $\mathbb{H}^s_{\exp}(\mathbb{K})$ can be identified with $\mathbb{H}^{-s}_{\exp}(\mathbb{K})$. The following proposition follows.

Proposition 2.2. *Let $s \in \mathbb{R}$. Then, for all $u \in \mathbb{H}^{-s}_{\exp}(\mathbb{K})$, there exists $v_s \in L^2(\mathbb{K})$ such that*

$$u = \sum_{k\in\mathbb{N}} \frac{(s/2)^k}{k!}\mathcal{F}(D^k_{\mathbb{K}} v_s).$$

Proof. Using the fact that $u \in \mathbb{H}^{-s}_{\exp}(\mathbb{K})$ we have

$$\tilde{u}_{-s}(\lambda,m) := \exp(-(s/2)\widetilde{N}(\lambda,m))\mathcal{F}u(\lambda,m) \in L^2(\widehat{\mathbb{K}}).$$

So $v_s = \mathcal{F}^{-1}(\tilde{u}_{-s}) \in L^2(\mathbb{K})$ and one can write

$$\mathcal{F}u(\lambda,m) = \exp((s/2)\widetilde{N}(\lambda,m))\tilde{u}_{-s}(\lambda,m) = \sum_{k\in\mathbb{N}} \frac{(s/2)^k}{k!}\widetilde{N}^k(\lambda,m)\mathcal{F}v_s(\lambda,m) =$$

$\sum_{k\in\mathbb{N}} \frac{(s/2)^k}{k!}\mathcal{F}(D^k_{\mathbb{K}} v_s)$. This finishes the proof. □

3. Generalized Pseudo-Differential Operators of Exponential Type

Definition 3.1. *Let $r \in \mathbb{R}$ and $l > 0$. A function $a : \mathbb{K} \times \hat{\mathbb{K}} \longrightarrow \mathbb{C}$ is said to be a symbol of exponential type and order (r,l) if a is in $C^\infty(\mathbb{K}\times\hat{\mathbb{K}})$ and for each $p \in \mathbb{N}$ there exists a constant $C > 0$ such that*

$$|D^p_{\mathbb{K}} a((x,t),(\lambda,m))| \leq C\exp(r\widetilde{N}(\lambda,m) - lN(x,t))$$

for all $((x,t),(\lambda,m)) \in \mathbb{K}\times\hat{\mathbb{K}}$.

We denote by $\mathbb{S}^{r,l}_{\exp}$ the class of all symbols of exponential type and order (r,l).

Using simple calculation one can verify that the symbol $a((x,t),(\lambda,m)) = e^{-lN^2(x,t)}$ belongs to $\mathbb{S}^{r,l}_{\exp}$.

Definition 3.2. *Let $r \in \mathbb{R}$ and $l > 0$. The pseudo-differential operator $A[(x,t),D]$ associated with a symbol $a \in \mathbb{S}^{r,l}_{\exp}$ is defined, for $u \in S_{*,\exp}$ by*

$$(A[(x,t),D]u)(x,t) = \mathcal{F}^{-1}(a((x,t),.)\mathcal{F}u(.))(x,t), \qquad for\ all\ (x,t) \in \mathbb{K}.$$

Theorem 3.1. (Main Theorem) *Let $r \in \mathbb{R}$, $l > 0$ and $A[(x,t),D]$ the pseudo-differential operator associated with a symbol $a \in \mathbb{S}^{r,l}_{\exp}$. Then for all $s \in \mathbb{R}_+$, $A[(x,t),D]$ maps continuously $\mathbb{H}^{r+s}_{\exp}(\mathbb{K})$ into $\mathbb{H}^{s}_{\exp}(\mathbb{K})$. In other terms there exists $C > 0$ such that for all $u \in \mathbb{H}^{r+s}_{\exp}(\mathbb{K})$ we have*

$$\|A[(x,t),D]u\|_{\mathbb{H}^{s}_{\exp}(\mathbb{K})} \leq C\|u\|_{\mathbb{H}^{r+s}_{\exp}(\mathbb{K})}.$$

To prove this theorem, we begin first by proving the following lemma.

Lemma 3.1. *Let $r \in \mathbb{R}$, $l > 0$ and $a \in \mathbb{S}^{r,l}_{\exp}$. Then the pseudo-differential operator $A[(x,t),D]$ associated with the symbol a satisfies*

$$\mathcal{F}\Big(A[(x,t),D]u\Big)(\lambda,m) = \int_{\hat{\mathbb{K}}} \mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}a(.,(\mu,n))\Big)(\lambda,m)\mathcal{F}u(\mu,n)d\gamma_\alpha(\mu,n).$$

Proof. Let us first prove that

$$\left(\int_{\hat{\mathbb{K}}} \mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}a(.,(\mu,n))\Big)(\lambda,m)\mathcal{F}u(\mu,n)d\gamma_\alpha(\mu,n)\right) \in L^1(\hat{\mathbb{K}}).$$

Let $\tau > 0$. Then it holds

$$\left|\exp(\tau\widetilde{N}(\lambda,m))\mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}a(.,(\mu,n))\Big)(\lambda,m)\right| =$$

$$\left|\sum_{k\in\mathbb{N}} \frac{\tau^k}{k!}\widetilde{N}^k((\lambda,m)\mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}a(.,(\mu,n))\Big)(\lambda,m)\right| \leq$$

$$\sum_{k\in\mathbb{N}} \frac{\tau^k}{k!}\left|\mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}D^k_{\mathbb{K}}a(.,(\mu,n))\Big)(\lambda,m)\right| =$$

$$\sum_{k\in\mathbb{N}} \frac{\tau^k}{k!}\left|\mathcal{F}\Big(\varphi_{(\mu,n)}(x,t)D^k_{\mathbb{K}}a(.,(\mu,n))\Big)(\lambda,m)\right| \leq$$

$$\sum_{k\in\mathbb{N}} \frac{\tau^k}{k!}\int_{\mathbb{K}}\Big|D^k_{\mathbb{K}}a((x,t),(\mu,n))\Big|d\mu_\alpha(x,t) \leq$$

$$\sum_{k\in\mathbb{N}} \frac{\tau^k}{k!} \exp(r\widetilde{N}(\mu,n)) \int_{\mathbb{K}} \exp(-lN(x,t))d\mu_\alpha(x,t) = C\exp(r\widetilde{N}(\mu,n)).$$

So, one can write $\left|\mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}a(.,(\mu,n))\Big)(\lambda,m)\right| \leq C\exp(r\widetilde{N}(\mu,n)-\tau\widetilde{N}(\lambda,m))$. And using the fact that $u \in S_{*,\exp} \subset \mathbb{H}^{\beta}_{\exp}(\mathbb{K})$, it holds, for $\beta > r$,

$$\int_{\hat{\mathbb{K}}} \left|\mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}a(.,(\mu,n))\Big)(\lambda,m)\mathcal{F}u(\mu,n)\right| d\gamma_\alpha(\mu,n) \leq$$

$$C\exp(-\tau\widetilde{N}(\lambda,m)) \int_{\hat{\mathbb{K}}} \exp((r-s)\widetilde{N}(\mu,n))d\gamma_\alpha(\mu,n) =$$

$$C\exp(-\tau\widetilde{N}(\lambda,m)) \in L^1(\hat{\mathbb{K}}).$$

On the other hand, by definition of $A[(x,t),D]$ and using Fubini's theorem, one has

$$(A[(x,t),D]u)(x,t) = \mathcal{F}^{-1}\Big(a((x,t),(\mu,n))\mathcal{F}u(.)\Big)(x,t) =$$

$$\int_{\hat{\mathbb{K}}} \varphi_{(\mu,n)}(x,t)a((x,t),(\mu,n))\mathcal{F}u(\mu,n)d\gamma_\alpha(\mu,n) =$$

$$\int_{\hat{\mathbb{K}}} \varphi_{(\mu,n)}(x,t)\mathcal{F}^{-1}\Big(\mathcal{F}a(.,.)(\mu,n)\Big)(x,t)\mathcal{F}u(\mu,n)d\gamma_\alpha(\mu,n) =$$

$$\int_{\hat{\mathbb{K}}} \mathcal{F}^{-1}\Big(\mathcal{T}^{(\alpha)}_{(\mu,n)}\mathcal{F}a(.,.)(\mu,n)\Big)(x,t)\mathcal{F}u(\mu,n)d\gamma_\alpha(\mu,n) =$$

$$\mathcal{F}^{-1}\left(\int_{\hat{\mathbb{K}}} \mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}a(.,.)(\mu,n)\mathcal{F}u(\mu,n)\Big)d\gamma_\alpha(\mu,n)\right)(x,t).$$

Applying the Fourier transform we get the result. □

Proof of Theorem 3.1. For $s > 0$ and $u \in S_{*,\exp}$, put

$$U_s(\lambda,m) := \exp((s/2)\widetilde{N}(\lambda,m))\mathcal{F}(A[(x,t),D]u)(\lambda,m).$$

Then, using Lemma 3.1, one has

$$U_s(\lambda,m) = \exp((s/2)\widetilde{N}(\lambda,m)) \int_{\hat{\mathbb{K}}} \mathcal{T}^{(\alpha)}_{(\mu,n)}\Big(\mathcal{F}a(.,(\mu,n))\Big)(\lambda,m)\mathcal{F}u(\mu,n)d\gamma_\alpha(\mu,n).$$

So it holds

$$\left|U_s(\lambda,m)\right| \leq \int_{\widehat{\mathbb{K}}} \exp(((s-\tau)/2)\widetilde{N}(\lambda,m)) \exp(r\widetilde{N}(\mu,n))(\lambda,m)\left|\mathcal{F}u(\mu,n)\right| d\gamma_\alpha(\mu,n)$$

$$= \int_{\widehat{\mathbb{K}}} \left[\exp(((s-\tau)/2)\widetilde{N}(\lambda,m) - (s/2)\widetilde{N}(\mu,n))\right] \times$$

$$\left[\exp(((r+s)/2)\widetilde{N}(\mu,n))(\lambda,m)\left|\mathcal{F}u(\mu,n)\right|\right] d\gamma_\alpha(\mu,n). \qquad (3.1)$$

On the other hand let us take $\tau = 2s$, then we have the following estimation

$$\exp(((s-\tau)/2)\widetilde{N}(\lambda,m) - (s/2)\widetilde{N}(\mu,n)) = \exp(-(s/2)(\widetilde{N}(\lambda,m) + \widetilde{N}(\mu,n))) \leq$$

$$\exp(-(s/2)|\lambda+\mu|) = (\mathcal{T}^{(\alpha)}_{(\lambda,m)}h)(\mu,n)$$

where $h(\lambda,m) = \exp(-(s/2)|\lambda|)$. Consequently

$$\left|U_s(\lambda,m)\right| \leq C \int_{\widehat{\mathbb{K}}} \left[(\mathcal{T}^{(\alpha)}_{(\lambda,m)}h)(\mu,n)\right] \times$$

$$\left[\exp(((r+s)/2)\widetilde{N}(\mu,n))\,|\mathcal{F}u(\mu,n)|\right] d\gamma_\alpha(\mu,n) \qquad (3.2)$$

and the right hand side of the inequality (3.1) is the convolution on $\widehat{\mathbb{K}}$ of the two $L^2(\widehat{\mathbb{K}})$ functions $h(\lambda,m) = \exp(-(s/2)|\lambda|)$ and

$$g(\lambda,m) = \exp(((r+s)/2)\widetilde{N}(\lambda,m))|\mathcal{F}u(\lambda,m).$$

Then $h \otimes g$ is in $L^2(\widehat{\mathbb{K}})$ and one has

$\|A[(x,t),D]u\|_{\mathbb{H}^s_{\exp}(\mathbb{K})} = \|U_s\|_{L^2(\widehat{\mathbb{K}})} \leq \|h \otimes g\|_{L^2(\widehat{\mathbb{K}})} \leq \|h\|_{L^2(\widehat{\mathbb{K}})}\|g\|_{L^2(\widehat{\mathbb{K}})} = C\|u\|_{\mathbb{H}^{r+s}_{\exp}(\mathbb{K})}$. This completes the proof of the theorem. □

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