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SEMISYMMETRY AND RICCI-SEMISYMMETRY FOR HYPERSURFACES OF SEMI-RIEMANNIAN SPACE FORMS

FILIP DEFEVER[1], RYSZARD DESZCZ, DOROTA KOWALCZYK
AND LEOPOLD VERSTRAELEN

ABSTRACT. In the context of P.J. Ryan's problem on the equivalence of the conditions $R \cdot R = 0$ and $R \cdot S = 0$ for hypersurfaces, we prove that there is indeed equivalence for hypersurfaces of semi-Riemannian space forms of any dimension, under an additional curvature condition of semisymmetric type.

1. INTRODUCTION

A semi-Riemannian manifold (M,g), dim $M \geq 3$, is called semisymmetric [15] if

$$R \cdot R = 0\,, \tag{1}$$

holds on M. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. A semi-Riemannian manifold (M,g), dim $M \geq 3$, is said to be Ricci-semisymmetric, if the following condition is satisfied

$$R \cdot S = 0\,. \tag{2}$$

Again, the class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. It is clear that every

[1]Postdoctoral Researcher F.W.O.-Vlaanderen, Belgium.
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semisymmetric manifold is Ricci-semisymmetric. The converse statement is however not true, as can be seen for instance from the material in [8].

Although the conditions (1) and (2) do not coincide for manifolds in general, it is a long standing question whether the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent for hypersurfaces of Euclidean spaces; cfr. Problem P 808 of [13] by P.J. Ryan, and references therein. Whereas for $n = 3$ this equivalence follows immediately, for $n > 3$ we have the following results. It had been proved in [14] that (1) and (2) are equivalent for hypersurfaces which have positive scalar curvature in a Euclidean space $\mathbb{E}^{n+1}$, $n > 3$. In [12] this result was generalized to hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$, $n > 3$, which have nonnegative scalar curvature and also to hypersurfaces of constant scalar curvature. [12] also proves that (1) and (2) coincide for hypersurfaces of Riemannian space forms with nonzero constant sectional curvature. Further, in [11] it was proved that (1) and (2) are equivalent for hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$, $n > 3$, under the additional global condition of completeness. In [4], it has been shown that the conditions (1) and (2) are equivalent for hypersurfaces of the Euclidean space $\mathbb{E}^5$. In [2] a negative answer to the above mentioned question was given for hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 5$. Indeed, [2] gives an example of a hypersurface M^5 of $\mathbb{E}^6$ which satisfies $R \cdot S = 0$, but which is not semisymmetric; this proves that both concepts are not equivalent for hypersurfaces of Euclidean spaces in general.

Although the fundamental question has now been solved, a number of new questions can be raised. Indeed, one may e.g. ask for a classification of the Ricci-semisymmetric hypersurfaces of the Euclidean spaces which are not semisymmetric. One can also consider the more general problem, whether (1) and (2) are equivalent for hypersurfaces of a semi-Riemannian space form $N^{n+1}(c)$. For example, [5] proves that there is indeed equivalence for all hypersurfaces of a 5-dimensional semi-Riemannian space form, thus generalizing the result of [4]; in [6] it was shown that (1) and (2) are equivalent for Lorentzian hypersurfaces of a Minkowski space $\mathbb{E}^{n+1}_1$, $n \geq 4$. [6] also proves that (1) and (2) are equivalent for para-Kähler hypersurfaces of a semi-Euclidean space

$\mathbb{E}_s^{2m+1}$, $m \geq 2$.

In order to tackle such questions, it is necessary to pursue more insight into the differences and look for an improved description and characterisation of the similarities of such hypersurfaces; one possibility for doing so is searching for sufficient conditions on hypersurfaces for both concepts (1) and (2) to be equivalent; at the same time, this narrows down the set of hypersurfaces where differences can occur. In this respect, [7] and [1] proved that (1) and (2) are equivalent for hypersurfaces of a semi- Euclidean space $\mathbb{E}_s^{n+1}$ which satisfy the curvature condition of pseudosymmetric type $C \cdot C = LQ(g, C)$, or the condition of semisymmetric type $C \cdot R = 0$, respectively. In the present paper, we generalize this latter result to all semi-Riemannian ambient spaces of constant sectional curvature; more precisely:

Theorem 1.1. *For hypersurfaces of a semi-Riemannian space form $\tilde{N}^{n+1}(c)$, $n \geq 4$, which satisfy the curvature condition $C \cdot R = 0$, the conditions of semisymmetry and Ricci-semisymmetry are equivalent.*

2. DEFINITIONS AND SOME USEFUL RELATIONS

Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^∞ and let ∇ be its Levi-Civita connection. We define on M the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$ and $\mathcal{C}(X, Y)$ by

$$\begin{aligned} (X \wedge_A Y)Z &= A(Y, Z)X - A(X, Z)Y\,, \\ \mathcal{R}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,, \\ \mathcal{C}(X, Y) &= \mathcal{R}(X, Y) - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right), \end{aligned}$$

where the Ricci operator $\mathcal{S}$ is defined by $S(X, Y) = g(X, \mathcal{S}Y)$, S is the Ricci tensor, κ the scalar curvature, A a symmetric $(0, 2)$-tensor and $X, Y, Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields of M. Next, we define the tensor G, the Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M, g) by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4)\,,$$

$$\begin{aligned} R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4)\,, \\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4)\,. \end{aligned}$$

For a $(0,k)$-tensor T, $k \geq 1$, and a symmetric $(0,2)$-tensor A, we define the $(0,k+2)$-tensors $R \cdot T$ and $Q(A,T)$ by

$$\begin{aligned} & (R \cdot T)(X_1,\ldots,X_k;X,Y) = (\mathcal{R}(X,Y) \cdot T)(X_1,\ldots,X_k) \\ = \; & -T(\mathcal{R}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{R}(X,Y)X_k)\,, \\ & Q(A,T)(X_1,\ldots,X_k;X,Y) = ((X \wedge_A Y) \cdot T)(X_1,\ldots,X_k) \\ = \; & -T((X \wedge_A Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},(X \wedge_A Y)X_k)\,. \end{aligned}$$

For (0,2)-tensors A and B we define its Kulkarni-Nomizu product $A \wedge B$ by

$$\begin{aligned} (A \wedge B)(X_1,X_2,X_3,X_4) = \; & A(X_1,X_4)B(X_2,X_3) + A(X_2,X_3)B(X_1,X_4) \\ & -A(X_1,X_3)B(X_2,X_4) - A(X_2,X_4)B(X_1,X_3)\,. \end{aligned}$$

Putting in the above formulas $T = R$, $T = S$, $T = C$ or $T = G$ and $A = g$ or $A = S$, we obtain the tensors $R \cdot R$, $R \cdot S$, $R \cdot C$, $Q(g,R)$, $Q(g,C)$, $Q(S,R)$, and $Q(S,C)$ respectively. The tensors $C \cdot R$ and $C \cdot C$ we define in the same way as the tensor $R \cdot R$; the tensor $C \cdot S$ is defined in the same way as the tensor $R \cdot S$. The $(0,2)$-tensor S^2 is defined by $S^2(X,Y) = S(\mathcal{S}X,Y)$, where $X,Y \in \Xi(M)$.

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be semisymmetric [15] if

(3) $$R \cdot R = 0$$

holds on M. Curvature conditions involving tensors of the form $R \cdot T$ only are called curvature conditions of semisymmetric type. Examples are the Ricci-semisymmetric space ($R \cdot S = 0$), and the Weyl semisymmetric spaces ($R \cdot C = 0$).

Manifolds satisfying curvature conditions involving tensors of both the form $R \cdot T$ and $Q(A,T)$ are called manifolds of pseudosymmetric type.

For example, we have semi-Riemannian manifolds (M,g), $n \geq 4$ satisfying at every point the following condition

$(*)$ the tensors $C \cdot R$ and $Q(g,C)$ are linearly dependent;

the condition $(*)$ is satisfied on a manifold (M,g) if and only if

$$C \cdot R = L\,Q(g,C) \tag{4}$$

holds on $U_C = \{x \in M \,|\, C \neq 0 \text{ at } x\}$, where L is a function on U_C. Other examples are the manifolds with pseudosymmetric Weyl tensor ($C \cdot C = LQ(g,C)$), and the Ricci-generalized pseudosymmetric manifolds ($R \cdot R = Q(S,R)$). For more information on the geometric motivation for the introduction of the concept of pseudosymmetry and a survey of various properties, including also applications to the general theory of relativity, we refer to the papers [8] and [16].

3. PROOF OF THE TECHNICAL RESULTS

We start with a couple of propositions of more general scope. In Section 4, we apply them to our situation, and draw the conclusions. Let (M,g), $n \geq 4$, be a semi-Riemannian manifold; we define the subset $U \subset M$ by $U = \{x \in M \,|\, S \neq 0 \text{ and } C \neq 0 \text{ at } x\}$. We note that $U \subset U_C$ and $U_S \cap U_C \subset U$. For the proofs of these propositions, we will however rely on a number of formulas which we collect in the following Lemma.

Lemma 3.1. *For a semi-Riemannian manifold (M,g), $n \geq 4$, satisfying $C \cdot R = L\,Q(g,C)$ on U_C, the following relations hold on U_C:*

$$R \cdot S = \frac{1}{n-2} Q(g,D)\,, \tag{5}$$

$$(C \cdot R)_{hijklm} = (R \cdot R)_{hijklm} - \frac{1}{n-2} Q(S,R)_{hijklm}$$

$$
\begin{aligned}
&+\frac{\kappa}{(n-1)(n-2)} Q(g,R)_{hijklm} \\
&-\frac{1}{n-2}\left(g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk}\right. \\
&\qquad \left. + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}\right), \qquad (6)
\end{aligned}
$$

$$A_{hijk} + A_{ihjk} = \frac{1}{n-2}\left(g_{hj}D_{ik} + g_{ij}D_{hk} - g_{hk}D_{ij} - g_{ik}D_{hj}\right), \qquad (7)$$

$$g^{hm}Q(S,R)_{hijklm} = -A_{iljk} - \kappa\, R_{lijk} + S_{kl}S_{ij} - S_{jl}S_{ik}, \qquad (8)$$

$$g^{hm}Q(g,C)_{hijklm} = -(n-1)\, C_{lijk}, \qquad (9)$$

$$B_{ij} \;=\; S^{rs}R_{rijs} \;=\; -\frac{1}{n-2}\left(S^2_{ij} - \kappa\, S_{ij}\right), \qquad (10)$$

$$A_{mijk} = S_m^{\;p} R_{pijk}, \qquad (11)$$

where $S_m^{\;p} = g^{rp}S_{mr}$ *and* D_{ij} *are the local components of the* $(0,2)$*-tensor* D*, defined by*

$$D = S^2 - \frac{\kappa}{n-1}\, S. \qquad (12)$$

Proof. First of all, from Proposition 3.1 of [1] we know that under the assumptions of the lemma, the condition (5) is satisfied on U_C. Writing the $(0,6)$-tensor $C \cdot R$ explicitly in components, we obtain (6). From (5), by (11), we get (7). Next, summing (7) cyclically in h, j, k we obtain

$$A_{hijk} + A_{jikh} + A_{kijh} = 0. \qquad (13)$$

Contracting now $Q(S,R)_{hijklm}$ and $Q(g,C)_{hijklm}$ with g^{hm} and applying (13) we obtain (8) and (9), respectively. Furthermore, contracting (7) with g^{hk} and using (11) and $S^{rs} = g^{rp}S_p^{\;s}$, we get (10). Our lemma is thus proved.

Proposition 3.1. *Let* (M,g)*,* $n \geq 4$*, be a semi-Riemannian manifold satisfying* $C \cdot R = L\, Q(g,C)$ *and*

$$R \cdot R - Q(S,R) = L_2\, Q(g,C), \qquad (14)$$

simultaneously on U_C*. Then, on* U*, the function* L *vanishes and*

$$C \cdot R = 0. \qquad (15)$$

Proof. Let the function L be nonzero at a point $x \in U$. First, remark that under the present assumptions the conditions for Proposition 3.1 of [1] are satisfied; consequently, the following relations hold on U_C:

$$C \cdot S = 0, \tag{16}$$

$$C \cdot C = L\,Q(g,C). \tag{17}$$

We note that (4) and (16) can be presented in the following form

$$(C \cdot R)_{hijklm} = L\,Q(g,C)_{hijklm}, \tag{18}$$

$$S_h^{\ p} C_{pijk} + S_i^{\ p} C_{phjk} = 0, \tag{19}$$

respectively. From (18) we get

$$\begin{aligned}(C \cdot R)_{hijklp} S_m^{\ p} + (C \cdot R)_{hijkmp} S_l^{\ p} = {} & L\,(Q(g,C)_{hijklp} S_m^{\ p} \\ & + Q(g,C)_{hijklm} S_l^{\ p}),\end{aligned} \tag{20}$$

Writing (20) out in components, and making use of (16), yields

$$\begin{aligned} & S_{hl} C_{mijk} + S_{il} C_{hmjk} + S_{jl} C_{himk} + S_{kl} C_{hijm} \\ & + S_{hm} C_{lijk} + S_{im} C_{hljk} + S_{jm} C_{hilk} + S_{km} C_{hijl} \\ & - g_{hm} S_l^{\ p} C_{pijk} + g_{im} S_l^{\ p} C_{phjk} - g_{jm} S_l^{\ p} C_{pkhi} + g_{km} S_l^{\ p} C_{pjhi} \\ & - g_{hl} S_m^{\ p} C_{pijk} + g_{il} S_m^{\ p} C_{phjk} - g_{jl} S_m^{\ p} C_{pkhi} + g_{kl} S_m^{\ p} C_{pjhi} = 0. \end{aligned} \tag{21}$$

Further, from (19) it follows easily that

$$S^{pq} C_{hpqk} = 0 \quad \text{and} \quad S_h^{\ p} C_{pijk} + S_j^{\ p} C_{pikh} + S_k^{\ p} C_{pihj} = 0. \tag{22}$$

Now, remark also that under the present assumptions the conditions for Proposition 3.2 of [1] are satisfied; consequently, the following relations hold on U_C:

$$R \cdot S = 0, \tag{23}$$

$$(a)\ S^2 = \frac{\kappa}{n-1} S, \qquad (b)\ \operatorname{tr}(S^2) = \frac{\kappa^2}{n-1}. \tag{24}$$

Furthermore, using (24), we get

$$S_h^{\ p} C_{pijk} = S_h^{\ p} R_{pijk} - \frac{1}{n-2} (S_{hk} S_{ij} - S_{hj} S_{ik}). \tag{25}$$

Contracting now (21) with g^{lh} and applying (19) and (22) we find

$$S_m^{\ p} C_{pijk} = \frac{\kappa}{n} C_{mijk} , \tag{26}$$

which, by transvection with $S_h^{\ m}$ and making use of (24), yields

$$\kappa \, S_h^{\ p} C_{pijk} = 0 . \tag{27}$$

From (26) and (27) it follows that

$$S_h^{\ p} C_{pijk} = 0 , \tag{28}$$

$$\kappa = 0 , \tag{29}$$

hold at x. Now (25) reduces to

$$S_h^{\ p} R_{pijk} = \frac{1}{n-2} (S_{hk} S_{ij} - S_{hj} S_{ik}) . \tag{30}$$

Applying in (6) the relations (4) and (14) we obtain

$$\begin{aligned} -\frac{n-3}{n-2} Q(S,R)_{hijklm} &= \frac{\kappa}{(n-1)(n-2)} Q(g,R)_{hijklm} \\ &\quad + (L_2 - L)\, Q(g,C)_{hijklm} \\ &\quad - \frac{1}{n-2} (g_{hl} A_{mijk} - g_{hm} A_{lijk} - g_{il} A_{mhjk} + g_{im} A_{lhjk} \\ &\quad + g_{jl} A_{mkhi} - g_{jm} A_{lkhi} - g_{kl} A_{mjhi} + g_{km} A_{ljhi}) . \end{aligned} \tag{31}$$

Contracting (31) with g^{hm} and using (8) and (9), and in view of (7), yields

$$\begin{aligned} -2(n-2)\, A_{lijk} &= -\frac{n-3}{n-2} Q(g,D)_{lijk} - (n-2)\kappa\, R_{lijk} \\ &\quad -(n-2)(n-1)(L_2 - L)\, C_{lijk} + (n-3)\,(S_{lk} S_{ij} - S_{jl} S_{ik}) \\ &\quad + \frac{\kappa}{n-1} (g_{lk} S_{ij} - g_{jl} S_{ik}) + g_{jl} B_{ik} - g_{hl} B_{ij} . \end{aligned} \tag{32}$$

Let now $x \in U_S \subset M$. Using (10) and (24) we get

$$\begin{aligned} \frac{\kappa}{n-1} S_{ij} - B_{ij} &= \frac{\kappa}{n-1} S_{ij} + \frac{1}{n-2} S^2_{ij} - \frac{\kappa}{n-2} S_{ij} \\ = \frac{1}{n-2} \left(S^2_{ij} - \frac{\kappa}{n-1} S_{ij} \right) &= \frac{1}{n(n-2)} \left(\mathrm{tr}(S^2) - \frac{\kappa^2}{n-1} \right) g_{ij} = 0 . \end{aligned}$$

Therefore (32) reduces to

$$(33)\quad A_{lijk} = \frac{\kappa}{2} R_{lijk} + \frac{n-1}{2}(L_2 - L)\, C_{lijk} - \frac{n-3}{2(n-2)} (S_{ij}S_{lk} - S_{lj}S_{ik})\,.$$

Next, comparing the right sides of (30)(a) and (33) and using (29) we obtain

$$(34)\qquad S_{hk}S_{ij} - S_{hj}S_{ik} = \frac{L_2 - L}{n-2}\, C_{hijk}\,.$$

From this, by making use of (16), we obtain

$$(35)\qquad (L_2 - L)\, C \cdot C = 0\,,$$

whence, by (17),

$$(L_2 - L)\, L\, Q(g, C) = 0\,.$$

Since $Q(g, C)$ is nonzero at x, the last equality implies

$$(36)\qquad L_2 = L\,.$$

Thus (34) reduces to $\operatorname{rank} S \leq 1$. Since $x \in U$, from the last inequality it follows that

$$(37)\qquad \operatorname{rank} S = 1\,.$$

Now (30), by (37) and (11), gives $A_{hijk} = 0$. Applying the last relation, (29) and (36) in (31) we obtain $Q(S, R) = 0$, which, in view of Proposition 4.1 of [3], implies $R \cdot R = Q(S, R) = 0$. Using this in (14) we get $L_2\, Q(g, C) = 0$, whence $L_2 = 0$. Now from (36) it follows that the function L vanishes at x, a contradiction. Our proposition is thus proved.

Proposition 3.2. *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold satisfying $R \cdot R - Q(S, R) = L_2\, Q(g, C)$ and $C \cdot R = 0$ on U.*

(i) *If $\kappa + (n-1)\, L_2 = 0$ is satisfied at a point $x \in U$ then $\kappa = 0$, $L_2 = 0$ and $\operatorname{rank} S = 1$ holds at x.*

(ii) *If $\kappa + (n-1)\, L_2 \neq 0$ is satisfied at a point $x \in U$ then $R \cdot R = 0$ holds at x.*

Proof. (33), by the assumption $L = 0$, takes the form

$$g^{pq} S_{hp} R_{qijk} = \frac{\kappa}{2} R_{hijk} + \frac{n-1}{2} L_2 C_{hijk} - \frac{n-3}{2(n-2)} (S_{hk} S_{ij} - S_{hj} S_{ik}) \,. \tag{38}$$

From (38) we obtain

$$\begin{aligned} & g^{pq}(R \cdot S)_{hplm} R_{qijk} + g^{pq} S_{hp} (R \cdot R)_{qijklm} \\ & - \frac{n-1}{2} L_2 (R \cdot C)_{hijklm} - \frac{\kappa}{2} (R \cdot R)_{hijklm} \\ & = - \frac{n-3}{2(n-2)} (S_{ij}(R \cdot S)_{hklm} + S_{hk} (R \cdot S)_{ij} \\ & - S_{ik} (R \cdot S)_{hjlm} - S_{hj} (R \cdot S)_{iklm}) \,. \end{aligned} \tag{39}$$

This, by making use of (23), $R \cdot C = R \cdot R$ and (14), reduces to

$$\begin{aligned} & S_h^{\ p} Q(S,R)_{pijklm} + L_2 \, S_h^{\ p} Q(g,C)_{pijklm} - \frac{n-1}{2} L_2^2 \, Q(g,C)_{hijklm} \\ = \ & \frac{\kappa}{2} Q(S,R)_{hijklm} + \frac{\kappa}{2} L_2 \, Q(g,C)_{hijklm} + \frac{n-1}{2} L_2 \, Q(S,R)_{hijklm} \,, \end{aligned} \tag{40}$$

whence, by symmetrization in h, i, we get

$$\begin{aligned} & S_h^{\ p} Q(S,R)_{pijklm} + S_i^{\ p} Q(S,R)_{phjklm} \\ + \ & L_2 \, S_h^{\ p} Q(g,C)_{pijklm} + L_2 \, S_i^{\ p} Q(g,C)_{phjklm} \ = \ 0 \,. \end{aligned}$$

This, by making use of the definitions of the tensors $Q(S,R)$ and $Q(g,C)$, (23) and (15), turns into

$$\begin{aligned} & S_{hl} T_{mijk} + S_{il} T_{mhjk} + L_2 \, g_{hl} B_{imjk} + L_2 \, g_{il} B_{hmjk} \\ = \ & S_{hm} T_{lijk} + S_{im} T_{lhjk} + L_2 \, g_{hm} B_{iljk} + L_2 \, g_{im} B_{hljk} \,, \end{aligned} \tag{41}$$

where $T_{mijk} = \frac{\kappa}{n-1} R_{mijk} - A_{mijk} + L_2 C_{mijk}$ and $B_{mijk} = S_{mp} g^{pq} C_{qijk}$. Now, symmetrizing (41) in the pairs h, i and l, m we get

$$\begin{aligned} & S_{hl} T_{imjk} + S_{mh} T_{iljk} + L_2 \, g_{hl} B_{mijk} + L_2 \, g_{mh} B_{lijk} \\ = \ & S_{li} T_{hmjk} + S_{mi} T_{hljk} + L_2 \, g_{li} B_{mhjk} + L_2 \, g_{mi} B_{lhjk} \,. \end{aligned} \tag{42}$$

We notice that, by making use of (23) and (15), we have

$$T_{mijk} \ = \ -T_{imjk} \,, \qquad B_{mijk} \ = \ -B_{imjk} \,. \tag{43}$$

Comparing the left sides of (41) and (42) and using the symmetry relations (43) we find

$$S_{li}T_{mhjk} - S_{mh}T_{lijk} = L_2\left(g_{li}B_{mhjk} - g_{hm}B_{lijk}\right). \tag{44}$$

Further, symmetrizing (44) in l, i we obtain

$$S_{li}T_{mhjk} = L_2\, g_{li}B_{mhjk}\,, \tag{45}$$

which, by transvection with $S_{hp}g^{pl}$ and making use of (24)(a), yields

$$\frac{\kappa}{n-1}\,T_{mhjk} = L_2\,B_{mhjk}\,.$$

Substituting this in (45) we get

$$\left(S_{li} - \frac{\kappa}{n-1}\,g_{li}\right)T_{mhjk} = 0\,. \tag{46}$$

Since at every point of U the Ricci tensor S is nonzero, (46) implies $T_{lijk} = 0$, i.e.

$$\frac{\kappa}{n-1}\,R_{mijk} - A_{mijk} + L_2\,C_{mijk} = 0\,,$$

whence, by (38), we obtain

$$\kappa\,R_{mijk} = -(n-1)\,L_2\,C_{mijk} + \frac{n-1}{n-2}\left(S_{mk}S_{ij} - S_{mj}S_{ik}\right).$$

This, by an application of the definition of the Weyl tensor C, turns into

$$\begin{aligned}(\kappa + (n-1)\,L_2)\,R_{mijk} &= \frac{n-1}{n-2}\,L_2\,(g\wedge S)_{mijk}\\ &\quad -\frac{\kappa}{n-2}\,L_2\,G_{mijk} + \frac{n-1}{n-2}\left(S_{mk}S_{ij} - S_{mj}S_{ik}\right).\end{aligned} \tag{47}$$

(i) We now consider the first subcase; suppose that

$$L_2 = -\frac{\kappa}{n-1} \tag{48}$$

holds at a point $x \in U$. Now (47) reduces to

$$\frac{1}{2}\,S\wedge S = -L_2\,g\wedge S + \frac{\kappa}{n-1}\,L_2\,G\,,$$

which turns into

$$\left(S - \frac{\kappa}{n-1} g\right) \wedge \left(S - \frac{\kappa}{n-1} g\right) = 0 .$$

The last relation implies

$$S = \frac{\kappa}{n-1} g + \beta \omega \otimes \omega , \quad \beta \in I\!R , \quad \omega \in T^*_x(M) , \tag{49}$$

holds at x. From (49) we get

$$\frac{n-2}{n-1}\kappa = \beta \, tr(\omega \otimes \omega) , \tag{50}$$

$$\mathcal{S}\omega = \left(\frac{\kappa}{n-1} + \beta \, tr(\omega \otimes \omega)\right) \omega , \tag{51}$$

$$S^2 = \frac{\kappa}{n-1} S + \beta \, \mathcal{S}\omega \otimes \omega . \tag{52}$$

Since ω is a nonzero covector, (52), together with (24)(a), implies $\mathcal{S}\omega = 0$. Now (51) reduces to

$$\frac{\kappa}{n-1} = -\beta \, tr(\omega \otimes \omega) .$$

This, together with (50), implies $\kappa = 0$, and, in a consequence, (48) reduces to $L_2 = 0$.

(ii) For the second subcase, we assume that

$$L_2 \neq -\frac{\kappa}{n-1} \tag{53}$$

holds at a point $x \in I\!R$. From (47), in view of Theorem 4.2 of [9], it follows that the tensor $R \cdot R$ vanishes at x, which completes the proof in this case.

Our proposition is thus proved.

4. CONCLUSIONS

The results of Section 3 allow us to draw the following conclusions.

Theorem 4.1. *Let M be a hypersurface of a semi-Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If the relation $C \cdot R = L\,Q(g, C)$ is satisfied on U then the function L vanishes on U.*

Proof. It is well known [10] that a hypersurface M of a semi-Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, fulfils (14). More precisely,

$$R \cdot R - Q(S, R) = -\frac{n-2}{n(n+1)}\,\tilde{\kappa}\,Q(g, C) \tag{54}$$

holds on M, where $\tilde{\kappa}$ is the scalar of the ambient space. Thus we see that under the assumption that M satisfies $C \cdot R = LQ(g, C)$ on U, the conditions of Proposition 3.1 are satisfied. Hence there follows that $L = 0$ on U.

Theorem 4.2. *Let M be a hypersurface of a semi-Riemannian space of nonzero constant curvature $N^{n+1}(c)$, $n \geq 4$. If the relation $C \cdot R = 0$ is satisfied on U then $R \cdot R = 0$ holds on U.*

Proof. Indeed, in view of [10] we have again that (54) is satisfied on M. Hence the conditions of Proposition 3.2 are fulfilled on U. Since we assume that the scalar curvature of the ambient space $N^{n+1}(c)$ is nonzero, and $L_2 = -\frac{n-2}{n(n+1)}\tilde{\kappa}$ here, case (i) of Proposition 3.2 cannot occur. We are thus necessarilly in the situation (ii), and $R \cdot R = 0$ holds on U.

This leads immediately to the main Theorem.

Theorem 4.3. *For hypersurfaces a semi-Riemannian space form $\tilde{N}^{n+1}(c)$ which satisfy the curvature condition $C \cdot R = 0$, the conditions of semisymmetry and Ricci-semisymmetry are equivalent.*

Proof. Since $R \cdot R = 0$ always implies $R \cdot S = 0$, we only have to proof that, under the additional condition $C \cdot R = 0$, Ricci-semisymmetry indeed leads to semisymmetry. In case the ambient space is a semi-Riemannian space with zero constant curvature, and thus a semi-Euclidean space, the above equivalence has been established in Theorem 3.2. When the ambient space is a semi-Riemannian space with nonzero constant curvature, the requested implication follows from Theorem 3.2 on the set U. Therefore, there remains only to sort out what happens on hypersurfaces of a semi-Riemannian space with (nonzero) constant curvature, at points where either $C = 0$ or $S = 0$. However, at points where $C = 0$, $R \cdot S = 0$ is always equivalent to $R \cdot R = 0$, and at points where $S = 0$, $C \cdot R = 0$ implies $R \cdot R = 0$; we thus recover semisymmetry also in the remaining situations.

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Ryszard Deszcz, Dorota Kowalczyk
Department of Mathematics
Agricultural University of Wrocław
ul. Grunwaldzka 53
PL - 50-357 Wrocław
Poland

rysz@ozi.ar.wroc.pl

Filip Defever, Leopold Verstraelen
Departement Wiskunde
Katholieke Universiteit Leuven
Celestijnenlaan 200 B
3001 Heverlee
Belgium

Filip.Defever@wis.kuleuven.ac.be

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