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CLOSE-TO-CONVEX FUNCTIONS OF COMPLEX ORDER RELATED TO RUSCHEWEYH DERIVATIVE

FATIMA M. AL-OBOUDI

ABSTRACT. Let $\mathcal{A}$ denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the unit disk E. Singh and Singh introduced the class R_n of functions $f \in \mathcal{A}$ satisfying $Re\{z(D^n f(z))'/D^n f(z)\} > 0$, where D^n is the Ruscheweyh derivative defined as the convolution operator
$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$. We introduce the class $K_n^b[A, B]$ of functions $f \in \mathcal{A}$, satisfying, for $z \in E$,

$$1 + \frac{1}{b}\left\{\frac{z(D^n f(z))'}{D^n g(z)1} - 1\right\} \prec \frac{1+Az}{1+Bz}$$

for some $g \in R_n$, where $\prec$ denote subordination, $b \neq 0$ is any complex number and A and B are arbitrary fixed numbers, $-1 \leq B < A \leq 1$. Coefficient bounds, inclusion and convolution relations are studied, and the radius of the largest disk in which every $f \in K_n^b[A, B]$ belongs to $K_n^1[1, -1]$ is determined.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f which are analytic in the unit disk $E = \{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$.

An analytic function f on E is said to be subordinate to an analytic function g on E (written $f \prec g$) if $f(z) = g(w(z)), z \in E$, for some analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in E. The Hadamard

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product (convolution) of two power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

is defined as the power series

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad z \in E$$

Denote by $D^n : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$\begin{aligned} D^n f(z) &= \frac{z}{(1-z)^{n+1}} * f, \quad n \in N_0 = \{0, 1, 2, \ldots\} \\ &= z + \sum_{k=2}^{\infty} \frac{(k+n-1)!}{n!(k-1)!} a_k z^k. \end{aligned} \tag{1.1}$$

Notice that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$. The symbol $D^n f$ was named the Ruscheweyh derivative by Al-Amiri [2]. In recent years, many classes, defined by Ruscheweyh derivatives, were studied.

The class R_n, $n \in N_0$, introduced by Singh and Singh [8], is defined as follows. $f \in R_n$ if, and only if, $f \in \mathcal{A}$ and

$$Re \frac{z(D^n f(z))'}{D^n f(z)} > 0, \quad z \in E.$$

Note that $R_0 \equiv S^*$, the well-known class of starlike functions and $R_1 \equiv C$ is the class of convex functions. It is known [8] that $R_{n+1} \subset R_n$, $n \in N_0$, hence R_n consists of starlike functions.

The class $P[A, B]$ introduced by Janowski [5], is defined as follows. For real numbers A and B, $-1 \leq B < A \leq 1$, $p \in P[A, B]$ if, and only if, p is analytic on E, $p(0) = 1$ and

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in E.$$

Al-Amiri and Thotage [3] introduced the class $K(b)$ of functions close-to-convex of complex order $b, b \neq 0$ and its generalization, the classes $K_n(b)$. A function $f \in K(b)$ if, and only if,

$$Re\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{g(z)}-1\right)\right\}>0, \quad z \in E,$$

for some starlike function g.

We have the following.

Let $K_n^b[A,B]$ denote the classes of functions f in $\mathcal{A}$ satisfying the condition

$$1+\frac{1}{b}\left\{\frac{z(D^n f(z))'}{D^n g(z)}-1\right\} \prec \frac{1+Az}{1+Bz}, \quad z \in E \tag{1.2}$$

for some $g \in R_n$, where $-1 \leq B < A \leq 1$, $n \in N_0$, $b \neq 0, b$ complex. We notice that $K_0^b[1,-1] \equiv K(b), K_n^b[1,-1] \equiv K_n(b)$ and $K_0^1[1,-1] \equiv K$, the well-known class of close-to-convex functions.

In this paper we show that the functions in $K_n^b[A,B]$ are close-to-convex of complex order and we investigate certain properties of functions belonging to these classes.

2. INCLUSION AND CONVOLUTION RELATIONS

To derive our results we need the following:

Lemma 2.1 [6] *Let β, γ be complex numbers. Let $h \in \mathcal{A}$ be convex univalent in E with $h(0)=1$ and $Re[\beta h(z)+\gamma]>0, z \in E$ and $q \in \mathcal{A}$, with $q(z) \prec h(z), z \in E$. If $p(z)=1+p_1 z+\ldots$ is analytic in E then*

$$p(z)+\frac{zp'(z)}{\beta q(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 2.2 [7] *Let ϕ be convex and g starlike in E. Then for F analytic in E with $F(0)=1$, $\frac{\phi * Fg}{\phi * g}(E)$ is contained in the convex hull of $F(E)$.*

The following result shows that as n increases the classes $K_n^b[A,B]$ become smaller.

Theorem 2.1. *$K_{n+1}^b[A,B] \subset K_n^b[A,B]$ for $n \in N_0$.*

Proof. Let $f \in K_{n+1}^b[A,B]$. Then

$$1+\frac{1}{b}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)}-1\right\} \prec \frac{1+Az}{1+Bz} \tag{2.1}$$

for some $g \in R_{n+1}$. Set

$$1+\frac{1}{b}\left\{\frac{z(D^{n}f(z))'}{D^{n}g(z)}-1\right\} = p(z). \tag{2.2}$$

Since $R_{n+1} \subset R_n, g \in R_n$, we have to show that $p \prec \dfrac{1+Az}{1+Bz}$. From (2.2) we have

$$\frac{z(D^{n}f(z))'}{D^{n}g(z)} = b[p(z)-1]+1. \tag{2.3}$$

By using the known identity

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - nD^n f(z) \tag{2.4}$$

in (2.3) and then differentiating, we get

$$\begin{aligned} z(D^{n+1}f(z))' &= \frac{1}{n+1}z(D^n g(z))'\{b(p(z)-1)+1\} \\ &\quad +\frac{1}{n+1}D^n g(z)\left\{bzp'(z)+n\frac{z(D^n f(z))'}{D^n g(z)}\right\}. \end{aligned}$$

Using (2.3) we obtain

$$\begin{aligned} z(D^{n+1}f(z))' &= \frac{1}{n+1}z(D^n g(z))'\{b(p(z)-1)+1\} \\ &\quad +\frac{1}{n+1}D^n g(z)[\{bzp'(z)+n\{b(p(z)-1)+1\}]. \end{aligned}$$

Hence

$$\begin{aligned} \frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} &= \frac{1}{n+1}\frac{z(D^n g(z))'}{D^{n+1}g(z)}\{b(p(z)-1)+1\} \\ &\quad +\frac{1}{n+1}\frac{D^n g(z)}{D^{n+1}g(z)}[bzp'(z)+n\{b(p(z)-1)+1\}]. \end{aligned} \tag{2.5}$$

Applying (2.4) for the function g we obtain

$$\frac{z(D^n g(z))'}{D^{n+1}g(z)} = (n+1) - n\frac{D^n g(z)}{D^{n+1}g(z)}. \tag{2.6}$$

Using (2.6) in (2.5) we get

$$\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} = b(p(z)-1) + 1 + \frac{b}{n+1}\left(\frac{D^n g(z)}{D^{n+1}g(z)}\right) zp'(z).$$

Hence

$$1 + \frac{1}{b}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - 1\right\} = p(z) + \frac{1}{n+1}\left(\frac{D^n g(z)}{D^{n+1}g(z)}\right) zp'(z). \tag{2.7}$$

Since $g \in R_{n+1}$ and $R_{n+1} \subset R_n$ [8], this implies that $g \in R_n$. From the definition of R_n and (2.4) we have

$$Re\ \frac{D^{n+1}g(z)}{D^n g(z)} > \frac{n}{n+1}. \tag{2.8}$$

Using (2.8) in (2.7) with $q(z) = \frac{D^{n+1}g(z)}{D^n g(z)}$ we obtain

$$1 + \frac{1}{b}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - 1\right\} = p(z) + \frac{zp'(z)}{(n+1)q(z)}. \tag{2.9}$$

Since $f \in K^b_{n+1}[A,B]$, then using (2.1) in (2.9) we get

$$P(z) + \frac{zP'(z)}{(n+1)q(z)} \prec \frac{1+Az}{1+Bz}. \tag{2.10}$$

From (2.8) we see that $q(z) \prec \frac{1+Az}{1+Bz}$, hence applying Lemma 2.1 we obtain the required result.

In the following we show how to move to different classes of $K^b_n[A,B]$ through convolution with hypergeometric functions. Recall the generalized hypergeometric function

$${}_mF_n(\alpha_1,\alpha_2,\ldots,\alpha_m;\beta_1,\beta_2,\ldots,\beta_n;z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k\ldots(\alpha_m)_k}{(\beta_1)_k(\beta_2)_k\ldots(\beta_n)_k k!}z^k,$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)\dots(a+k-1)$ for $k \geq 1$. We will apply this operator after establishing the following lemma.

Lemma 2.3. *Let $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ be defined by*

$$\mathcal{L}(f) = \frac{n+1}{z^n} \int_0^z t^{n-1} f(t)dt.$$

Then $f \in K_n^b[A,B]$ if, and only if, $\mathcal{L}(f) \in K_{n+1}^b[A,B]$.

Proof. Let $g \in R_n$. Then $\mathcal{L}(g) \in R_{n+1}$ [1].

We have to show that

$$1 + \frac{1}{b}\left\{\frac{z(D^n f(z))'}{D^n g(z)} - 1\right\} \prec \frac{1+Az}{1+Bz}$$

if, and only if,

$$1 + \frac{1}{b}\left\{\frac{z(D^{n+1}\mathcal{L}(f))'}{D^{n+1}\mathcal{L}(g)} - 1\right\} \prec \frac{1+Az}{1+Bz}.$$

It is sufficient to show that $D^n f = D^{n+1}\mathcal{L}(f)$. In [1] it is shown that $D^n g(z) = D^{n+1}\mathcal{L}(g)$. Therefore, $D^n f(z) = D^{n+1}\mathcal{L}(f)$.

Theorem 2.2. *Let*

$$H(z) = {}_{m+1}F_m(n+1, \dots, n+1, 1 : n+2, n+2, \dots, n+2, z)$$

be a hypergeometric function. Then $f \in K_n^b[A,B]$ if, and only if, $(f * zH) \in K_{n+m}^b[A,B]$ for $m = 1, 2, \dots$.

Proof. For $f(z) = \sum_{k=1}^{\infty} a_k z^k \in \mathcal{A}$, $a_1 = 1$ we have

$$\begin{aligned}
\mathcal{L}(f) &= \sum_{k=1}^{\infty} \frac{n+1}{n+k} a_k z^k \\
&= \left(z \sum_{k=0}^{\infty} \frac{n+1}{n+k+1} z^k\right) * \sum_{k=1}^{\infty} a_k z^k \\
&= \left(z \sum_{k=0}^{\infty} \frac{(n+1)_k (1)_k}{(n+2)_k k!} z^k\right) * f(z) \\
&= [z\,{}_2F_1(n+1, 1, n+2; z)] * f(z),
\end{aligned}$$

belongs to $K_{n+1}^b[A,B]$. Applying Lemma 2.3 m times we get the required result.

The following result is the analogue of the Polya-Schoenberg conjecture [7] for functions belonging to $K_n^b[A,B]$.

Theorem 2.3. *Let $f \in K_n^b[A,B]$, $\phi \in C$. Then $(f * \phi) \in K_n^b[A,B]$.*

Proof. Since $f \in K_n^b[A,B]$, then

$$1 + \frac{1}{b}\left\{\frac{z(D^n f(z))}{D^n g(z)} - 1\right\} \prec \frac{1+Az}{1+Bz},$$

for some $g \in R_n$. Hence

$$\frac{z(D^n f(z))}{D^n g(z)} \prec \frac{1 + \{b(A-B)+B\}z}{1+Bz} = \tau(z) \quad \text{(say)}.$$

Now

$$\begin{aligned} \frac{z(D^n(f * \phi))}{D^n(g * \phi)} &= \frac{\phi * z(D^n f(z))}{\phi * D^n g(z)} \\ &= \frac{\phi * z\frac{(D^n f(z))'}{D^n g(z)} \cdot D^n g(z)}{\phi * D^n g(z)}. \end{aligned}$$

Since $g \in R_n$ implies that $D^n g \in S^*$ and since τ is convex in E, then applying Lemma 2.2 we deduce that

$$\frac{\phi * z\frac{(D^n f(z))'}{D^n g(z)} \cdot D^n g(z)}{\phi * D^n g(z)} \prec \tau(z).$$

This implies that

$$1 + \frac{1}{b}\left\{\frac{z(D^n(f * \phi))}{D^n(g * \phi)} - 1\right\} \prec \frac{1+Az}{1+Bz}.$$

Hence $(f*\phi) \in K_n^b[A,B]$, where $(g*\phi) \in R_n[1]$ and the proof is complete.

Corollary 2.1. Let $f \in K_n^b[A,B]$, $\phi(z) = \sum_{k=1}^{\infty} \frac{n+1}{n+k} z^k$. Then $\mathcal{L}(f) = (\phi * f) \in K_n^b[A,B]$

3. COEFFICIENT BOUNDS

To derive our result we need the following:

Lemma 3.1. [4] Let $p \in P[A,B]$ and $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$.

Then

$$|c_k| \leq (A-B).$$

Theorem 3.1. *Let $f \in K_n^b[A,B]$ be given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then*

$$|a_k| \leq \left\{1 + \frac{|b|}{2}(A-B)(k-1)\right\} \frac{n!(k-1)!}{(k+n-1)!} \tag{3.1}$$

This result is sharp.

Proof. Since $f \in K_n^b[A,B]$, there exists a function $g \in R_n$ such that

$$1 + \frac{1}{b}\left\{\frac{z(D^n f(z))'}{D^n g(z)} - 1\right\} = p(z), \quad p \in P[A,B].$$

Hence

$$z(D^n f(z))' = D^n g(z)\{b(p(z)-1)+1\}. \tag{3.2}$$

Let

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \quad \text{and} \quad D^n g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{3.3}$$

Using (3.3) in (3.2) we obtain

$$z + \sum_{k=2}^{\infty} k \frac{(k+n-1)!}{n!(k-1)!} a_k z^k = \left(z + \sum_{k=2}^{\infty} b_k z^k\right)\left(b \sum_{k=1}^{\infty} c_k z^k + 1\right).$$

Equating the coefficients of z^n in both sides we get

$$k\frac{(k+n-1)!}{n!(k-1)!} a_k = b(c_{k-1} + b_2 c_{k-2} + b_3 c_{k-3} + \cdots + b_{k-1} c_1) + b_k.$$

Hence

$$k\frac{(k+n-1)!}{n!(k-1)!}|a_k| \leq |b|(|c_{k-1}| + |b_2||c_{k-2}| + \cdots + |b_{k-1}||c_1|) + |b_k|. \tag{3.4}$$

Since $g \in R_n$ implies that $D^n g \in S^*$, using the well-known coefficient estimates for starlike functions we get $|b_k| \leq k$, using this and Lemma 3.1 in (3.4) we obtain

$$\begin{aligned} k\frac{(k+n-1)!}{n!(k-1)!}|a_k| &\leq |b|[(A-B)+2(A-B)+\cdots+(k-1)(A-B)]+k \\ &= |b|(A-B)(1+2+\cdots+k-1)+k \\ &= \frac{|b|}{2}(A-B)k(k-1)+k. \end{aligned}$$

Hence

$$|a_k| \leq \left\{1+\frac{|b|}{2}(A-B)(k-1)\right\}\frac{n!(k-1)!}{(k+n-1)!}.$$

The equality sign in (3.1) holds for the function f given by

$$(D^n f(z)) = \frac{1+\{b(A-B)-1\}z}{(1-z)^3}. \tag{3.5}$$

Let $g \in \mathcal{A}$ be defined so that $D^n g(z) = \dfrac{z}{(1-z)^2}$. The definition of R_n implies that $g \in R_n$. Simple calculations shows that

$$1+\frac{1}{b}\left\{\frac{z(D^n f(z))}{D^n g(z)}-1\right\} = \frac{1+(A-B-1)z}{1-z} \prec \frac{1+Az}{1+Bz}.$$

The function f defined in (3.5) has the power series representation in E

$$f(z) = z+\sum_{k=2}^{\infty}\frac{n!(k-1)!}{(k+n-1)!}\left\{1+\frac{b}{2}(A-B)(k-1)\right\}z^k.$$

4. UNIVALENCE

If we put $b=1$, $A=1$, $B=-1$ in Theorem 2.1, we have

$$K^1_{n+1}[1,-1] \subset K^1_n[1,-1] \subset \ldots \subset K^1_0[1,-1] \equiv K,$$

which means that functions in $K^1_n[1,-1]$ are close-to-convex and hence univalent.

It is an open problem to find the bounds for b in which the functions in $K_n^b[A,B]$ are univalent, but we can find the radius of the largest disk in which every $f \in K_n^b[A,B]$ belongs to $K_n^1[1,-1]$, and hence univalent.

We need the following:

Lemma 4.1 [5] *Let $p \in P[A,B]$. Then for $|z| = r < 1$*

$$\left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2}.$$

This result is sharp.

Theorem 4.1. *Let $f \in K_n^b[A,B]$. Then $f \in K_n^1[1,-1]$ for $|z| < r_0$ where*

$$(4.1)\quad r_0 = 2\left[|b|(A-B) + \sqrt{|b|^2(A-B)^2 + 4B[(A-B)Re\, b + B]}\right]^{-1}.$$

This result is sharp.

Proof. Since $f \in K_n^b[A,B]$, there exists a function $g \in R_n$ such that

$$1 + \frac{1}{b}\left\{ \frac{z(D^n f(z))'}{D^n g(z)} - 1 \right\} = p(z), \quad p \in P[A,B].$$

Using Lemma 4.1 we obtain

$$\left| 1 + \frac{1}{b}\left\{ \frac{z(D^n f(z))'}{D^n g(z)} - 1 \right\} - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2}.$$

Hence

$$(4.2)\qquad \left| \frac{z(D^n f(z))'}{D^n g(z)} - \frac{1-Br^2[(A-B)b+B]}{1-B^2r^2} \right| \le \frac{(A-B)r|b|}{1-B^2r^2}$$

From (4.2) we have

$$Re\, \frac{z(D^n f(z))'}{D^n g(z)} \ge \frac{1-(A-B)|b|r - B[(A-B)Re\, b + B]r^2}{1-B^2r^2},$$

where $|z| = r$. Thus $Re\,\dfrac{z(D^n f(z))}{D^n g(z)} > 0$ for $r < r_0$ where r_0 is given by (4.1).

To show that the result is sharp consider the function f defined by

$$(D^n f(z)) = \frac{1 + [b(A-B)+B]z}{(1-z)^2(1+Bz)}. \quad \text{and let} \quad D^n g(z) = \frac{z}{(1-z)^2} \quad \text{and}$$

$$u = -r\left(Br + \sqrt{\frac{\bar{b}}{b}}\right) \Big/ \left(1 + rB\sqrt{\frac{\bar{b}}{b}}\right). \quad \textit{Then}$$

$$u\frac{(D^n f(u))}{D^n g(u)} = \frac{1 - |b|(A-B)r - B[(A-B)b+B]r^2}{1 - B^2 r^2},$$

which has a zero real part at r given by (4.1).

REFERENCES

1. O. Ahuja and H. Silverman, *Function classes related to Ruscheweyh Derivatives*, J. Austral. Math. Soc. (Series A) **47** (1989), 438-444.

2. H. Al-Amiri, *On Ruscheweyh Derivative*, Ann. Polon. Math. **38** (1980), 87-94.

3. H. Al-Amiri and S. Thotage, *On close-to-convex functions of complex order*, Internat. J. Math. & Math. Sci. **13**(2) (1990), 321-330.

4. V. Anh, *K-fold symmetric starlike univalent functions*, Bull. Austral. Math. Soc. **32** (1985), 419-436.

5. I. Janowski, *Some extremal problem for certain families of analytic functions I*, Ann. Polon. Math. **28** (1973), 298-326.

6. K. Padmanabhan and Parvatham, *Some applications of differential subordinations*, Bull. Austral. Math. Soc. **32** (1985), 321-330.

7. S. Ruscheweyh and T. Sheil-Small, *Hadamard products of Schlicht functions and the Polya-Schoenberg conjecture*, Comment. Math. Helv. **48** (1973), 119-135.

8. S. Singh and S. Singh, *Integrals of certain univalent functions*, Proc. Amer. Math. Soc. **77** (3), (1979), 336-340.

DEPARTMENT OF MATHEMATICS, GIRLS COLLEGE OF EDUCATION, RIYADH, SAUDI ARABIA.

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