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ON THE DIOPHANTINE EQUATION $d_1x^2 + 4d_2 = y^n$

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ABSTRACT.The object of this paper is to prove the following Theorem:
The diophantine equation

$$d_1x^2 + 4d_2 = y^n,$$

where d_1, d_2, x, y, n are positive integers such that $(d_1, d_2) = (x, y) = (2, y) = 1, d_1, d_2$ are square -free integers, n is an odd integer > 3 and $(n, h) = 1$ where h is the class number of the field $K = Q\left(\sqrt{-d_1d_2}\right)$, has no solutions in (d_1, d_2, x, y, n).

1. INTRODUCTION

Let d_1, d_2, x, y, p be positive integers. Many special cases of the diophantine equation

$$(1) \qquad d_1x^2 + 4d_2 = y^p, (d_1x, d_2) = 1, \quad p \text{ prime}>3, (x, y) = 1, p \nmid h$$

where d_1, d_2 are square-free integers and h is the class number of the field $Q\left(\sqrt{-d_1d_2}\right)$, have been considered in the last few years. The first result regarding this equation is due to Nagell [9], who proved that when $d_1 = d_2 = 1$, then equation (1) has only the positive solutions $x = y = 2$, $p = 3$ and $x = 11$, $y = 5, p = 3$. Ljunggren [6] studied this equation in full generality and he has found interesting theorems concerning its solutions. Also Le Maohua [8] proved that if $p \geq 8.5.10^6$ then equation (1) has no solution with $2 \nmid y$. In [1] we proved that if d_1 is odd, $d_2 = 2^{2k}$ and $p \equiv 1 \pmod 4$, then equation (1) has no solution with x odd. Recently Luca [7] proved that when $d_1 = 1$, $d_2 = 3^a$, then (1) has no solution. In the interesting paper [3] Bugeaud and Shory

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studied equation (1) where y is a fixed odd integer k coprime with $d_1 d_2$ and they gave a necessary and sufficient conditions on d_1, d_2 and k under which this equation has at most $2^{w(k)-1}$ solutions where $w(k)$ denotes the number of distinct prime divisors of k. In this paper we solve equation (1) completely.

2. Preliminaries

We start by giving some important definitions.

Definitions

A ***Lehmer pair*** is a pair (α, β) of algebraic integers such that $(\alpha+\beta)^2$ and $\alpha\beta$ are non-zero co-prime rational integers and α/β is not a root of unity.Given a Lehmer pair (α, β)one defines the corresponding sequence of Lehmer numbers by

$$u_n(\alpha, \beta) = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } n \text{ is odd,} \\ \dfrac{\alpha^n - \beta^n}{\alpha^2 - \beta^2} & \text{if } n \text{ is even.} \end{cases}$$

A prime number p is a ***primitive divisr*** of $u_n(\alpha, \beta)$ if p divides u_n, but does not divides $\left(\alpha^2-\beta^2\right)^2 u_1 u_2 \ldots u_{n-1}$.

A Lehmer pair (α, β) such that $u_n(\alpha, \beta)$ has no primitive divisors will be called $n-$***defective Lehmer pair***.

Now we reproduce the following results for future use.

Lemma 2.1 ([2]). *For $n > 30$, the nth term of any Lehmer sequences has a primitive divisor.*

Lemma 2.2 ([10]). *Let n satisfy $6 < n \leq 30$. Then up to equivalence all parameters of n-defective Lehmer pairs are given as follows:*

i. $n = 7$, $(a,b) = (1,-7), (1,-19), (3,-5), (5,-7), (13,-3), (14,-22)$.
ii. $n = 9, (a,b) = (5,-3), (7,-1), (7,-5)$,
iii. $n = 13, (a,b) = (1,-7)$,
iv. $n = 14, (a,b) = (3,-13), (7,-1), (7,-5), (19,-1), (22,-14)$,
v. $n = 15, (a,b) = (7,-1), (10,-2), (1,-7), (3,-5), (5,-7)$,
vi. $n = 18, (a,b) = (1,-7), (3,-50), (5,-7)$,

vii. $n = 24, (a, b) = (3, -5), (5, -3)$,
viii. $n = 26, (a, b) = (7, -1)$,
ix. $n = 30, (a, b) = (1, -7), (2, -10)$.

3. MAIN RESULTS

Theorem 3.1. *The diophantine equation*

$$d_1x^2 + 4d_2 = y^n, \tag{2}$$

where d_1, d_2, x, y, n *are positive integers such that* $(d_1, d_2) = (x, y) = (2, y) = 1$, d_1, d_2 *are square -free integers, n is an odd integer>3 and* $(n, h) = 1$ *where h is the class number of the field* $K = Q\left(\sqrt{-d_1d_2}\right)$, *has no solutions in* (d_1, d_2, x, y, n).

Proof. Let (d_1, d_2, x, y, n) be a solution of (2). The impossibility of the equations $3x^2 + 4 = y^n$ and $x^2 + 12 = y^n$, with x odd has been proved in [1, Theorem 1.8] and [4] respectively. We may therefore assume $d_1d_2 \neq 3$. Now factorize (2) in the field K,

$$\left(x\sqrt{d_1} + 2\sqrt{-d_2}\right)\left(x\sqrt{d_1} - 2\sqrt{-d_2}\right) = y^n$$

The principal ideal $[x\ \sqrt{d_1} + 2\sqrt{-d_2}]$ and its conjugate ideal are co-prime, so

$$\left[x\sqrt{d_1} + 2\sqrt{-d_2}\right] = \pi^n\ ,$$

for some ideal π in K. It follows that π^n is principal and since $(n, h) = 1$, therefore π is principal ideal, say $\pi = [\xi]$ for some element ξ in K. So we get the equation

$$\left[x\sqrt{d_1} + 2\sqrt{-d_2}\right] = [\xi]^n,$$

and consequently

$$\left(x\sqrt{d_1} + 2\sqrt{-d_2}\right) = \varepsilon\xi^n,$$

for some ε in K. Therefore we have the following two cases:

$$x\sqrt{d_1} + 2\sqrt{-d_2} = \left(\frac{a\sqrt{d_1} + b\sqrt{-d_2}}{2}\right)^n, \quad a \equiv b \equiv 1 \pmod 2$$

$$x\sqrt{d_1} + 2\sqrt{-d_2} = \left(a\sqrt{d_1} + b\sqrt{-d_2}\right)^n,$$

for some rational integers a and b.

By equating the coefficients of $\sqrt{-d_2}$, we deduce that the first case is impossible.

Now we consider the second case. Again equating the coefficients of $\sqrt{-d_2}$, we get the relation

$$2 = \sum_{r=0}^{\frac{n-1}{2}} \binom{n}{2r+1} a^{n-1-2r} b^{2r+1} d_1^{\frac{n-1}{2}-r} \left(-d_2\right)^r , \tag{3}$$

such that $y = a^2 d_1 + b^2 d_2$, and $(ad_1, bd_2) = 1$. Since y is odd, therefore, a and b have the opposite parity. Let

$$\alpha = a\sqrt{d_1} + b\sqrt{-d_2}, \ \beta = a\sqrt{d_1} - b\sqrt{-d_2}.$$

Then we get

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{2}{b}. \tag{4}$$

It is easy to verify that (α, β) is a Lehmer pair. Further, let

$$u_t\left(\alpha, \beta\right) = \frac{\alpha^t - \beta^t}{\alpha - \beta}, \quad t \geq 0$$

be the corresponding sequence of Lehmer number.

By Waring's formula [5, Formula 1.76], we get

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \sum_{i=0}^{\frac{n-1}{2}} \begin{bmatrix} n \\ i \end{bmatrix} \left(-4b^2 d_2\right)^{\frac{n-1-2i}{2}} y^i,$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(n-i-1)!n}{(n-2i)!i!}, \quad i = 0, \ldots, \frac{n-1}{2},$$

are positive integers. This implies that $\left(\alpha^n - \beta^n\right) / \left(\alpha - \beta\right)$ is an odd integer.

So $b = \pm 2$, hence ad_1 is odd, and from (4) we get

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \pm 1. \tag{5}$$

It implies that $u_n\left(\alpha, \beta\right)$ has no primitive divisor. By Lemmas 2.1 and 2.2, we see that if (5) holds, then $n \leq 5$. So we have $n = 5$, then from (3), we get

$$\pm 2 = \sum_{r=0}^{2} \binom{5}{2r+1} a^{4-2r} 2^{2r+1} d_1^{2-r} \left(-d_2\right)^r ,$$

dividing this equation by 2, we get

$$\pm 1 = 5a^4d_1^2 - 40a^2d_1d_2 + 16d_2^2 \tag{6}$$

Considering equation (6) modulo 8, we get a contradiction. This completes the proof of our theorem. □

Now we consider the case $n = 3$.

Corollary 3.1. *The diophantine equation (1), may have a solution with x and n odd only when $n = 3$, d_1 odd and $3 \nmid d_2$. Furthermore the solution (if it exists) is given by $y = a^2d_1 + 4d_2$, where a satisfies $a^2 = (4d_2 \pm 1)/3d_1$.*

Proof. From the above theorem it sufficient to consider $n = 3$, then from (3) we get

$$\pm 1 = 3d_1a^2 - 4d_2,$$

From this relation we deduce that d_1 odd and $3 \nmid d_2$. □

Examples:

(1) Consider the diophantine equation $3x^2 + 28 = y^n$. Here $h = 4, (n, 4) = 1$ for all odd integers n, so from the Theorem, there is no solution for $n > 3$. If $n = 3$, then $a^2 = (28 \pm 1)/9$ which impossible. So the equation has no solution in odd integers x and n for all $n \geq 3$. This is first shown by Ljunggren in [6].

(2) Consider the diophantine equation $3x^2 + 8 = y^n$. Here $h = 2$, so from the Theorem, there is no solution for $n > 3$. If $n = 3$, then $a^2 = (8 \pm 1)/9 = 1$. Hence

$$y = a^2d_1 + 4d_2 = 1.3 + 4.2 = 11. \quad x = 21.$$

So the equation has a unique solution in odd integers x and n.

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