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THE INVARIANCE OF THE IDEAL CORE $(a_1 ..., a_n)$ OF A SET OF HOMOGENEOUS POLYNOMIALS

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ABSTRACT. The definition of the ideal *core* $(a_1, ..., a_n)$ of homogeneous polynomials $a_1, ..., a_n$ of positive degree in $R[x_1, ..., x_m]$ is in terms of the particular generators $x_1, ..., x_m$ of the polynomial ring. It is shown in this note that the ideal is in fact independent of these generators. Similarly core depends only on the ideal generated by $a_1, ..., a_n$ and not on the particular generating set.

1. INTRODUCTION

Let R be a commutative ring with unity, $x_1, ..., x_m$ indeterminates and $a_1, \ldots, a_n$ homogeneous elements (forms) of the polynomial ring $R[x_1, ..., x_m]$ with degrees $\rho_1, \ldots, \rho_n (> 0)$. In [3] we used the Koszul complex $K(a_1, \ldots, a_n;\ R[x_1, ..., x_m, x_1^{-1}, \ldots, x_m^{-1}])$ to construct a graded complex of free R-modules $\overline{K}(a_1, ..., a_n; R) = \oplus_{t \in \mathbb{Z}} K(a_1, \ldots, a_n; t; R)$ and denoted by *core* $(a_1, ..., a_n)$ the ideal of R that annihilates all the homology of $\overline{K}(a_1, ..., a_n; R)$. In the case of linear forms $a_i = \sum_{j=1}^{m} \alpha_{ij} x_j$ ($i = 1, ..., n$) the complexes first appeared in [2] as the complexes associated with the $n \times m$ matrix (α_{ij}) with *core* $(a_1, ..., a_n)$ the ideal generated by the $m \times m$ minors of (α_{ij}). Here this Fitting invariant is clearly independent of the choice of generators $x_1, \ldots, x_m$ of the symmetric algebra $R[x_1, ..., x_m]$; but for all other choices of the forms the complexes depend on the ordered set of generators, consequently so does *core* $(a_1, ..., a_n)$-apparently. The main purpose of this note is to show *core* $(a_1, ..., a_n)$ is invariant for invertible linear changes of $x_1, ..., x_m$. In addition the final section is

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devoted to showing that if the ideal $\sum_{i=1}^{n} a_i R[x_1, ..., x_m]$ contains that generated by forms $b_1, ..., b_p$, then *core* $(a_1, ..., a_n) \supseteq$ *core* $(b_1, ..., b_p)$.

In the case $m = n$ and the forms $a_1, ..., a_m$ are generic *core* $(a_1, ..., a_m)$ is generated by the classical resultant of $a_1, ..., a_m$ [4], so is known to be invariant (see [5; p.12] for example). A different approach to the resultant, still using the Koszul complex appears in the work of Gelfand, Kapranov and Zelevinsky [1]. They show it is the determinant of $K(a_1, ..., a_m; t; R)$ for each $t \in \mathbb{Z}$ (when R is a field). Now the invariance is immediate for $t > \sum_{i=1}^{m}(\rho_i - 1)$ as $K(a_1, ..., a_m; t; R)$ is the homogeneous component of degree t in the Koszul complex $K(a_1, ..., a_m;\ R[x_1, ..., x_m])$, and for $t < 0$ when $K(a_1, ..., a_m; t; R)$ is the degree t component of $K(a_1, ..., a_m; R[x_1^{-1}, ..., x_m^{-1}])$ and so the dual of $K(a_1, ..., a_m; \sum_{i=1}^{m}(\rho_i - 1) -t; R)$.

2. INVARIANCE FOR TRIANGULAR CHANGES OF VARIABLES

In this section we show that for homogeneous polynomials $a_1, ..., a_n$ of degree $\rho_1, ..., \rho_n > 0$ in $R[x_1, ..., x_m](= R[x])$ the ideal *core* $(a_1, ..., a_n)$ of R defined in terms of the generators $x_i (i = 1, ..., m)$ is the same as that defined in terms of the generators $X_i = \sum_{j=i}^{m} c_{ij} x_j$ $(i = 1, \ldots, m)$, where $c_{ii} (i = 1, ..., m)$ is a unit of R. We need some notation.

Let T_i be the set of products of linear polynomials $\sum_{j=i}^{m} r_j x_j$, where r_i is a unit of R, and define $V_0 \supseteq ... \supseteq V_m$ inductively by $V_m = R[x]$ and V_i is the ring of quotients $(V_{i+1})_{T_{i+1}}$. Similarly put $V_{j-1j} = V_j$ and inductively $V_{ij} = (V_{i+1j})_{T_{i+1}}$ for $i = j-2, \ldots, 0$ so $V_{0j} \supseteq \ldots \supseteq V_{j-1j} = V_j$ and V_{ij} is a subring of V_i for $i < j$. Further as an R-submodule of V_i we pick out U_i generated by all power products $x_1^{\rho_1} ... x_m^{\rho_m}$ with $\rho_1, ..., \rho_i \geq 0$ and $\rho_{i+1}, \ldots, \rho_m < 0 (i = 0, \ldots, m)$ i.e. $U_i = x_{i+1}^{-1} ... x_m^{-1} R[x_1, \ldots, x_i, x_{i+1}^{-1}, ..., x_m^{-1}]$.

Lemma 2.1. *$V_i = U_i \oplus \left(\sum_{j=i+1}^{m} V_{ij}\right)$ as R-modules $(i = 0, ..., m)$.*

Proof. By Induction on $m - i$ we first show $V_i = U_i + \sum_{j=i+1}^{m} V_{ij}$. The cases $i = m, m-1$ are immediate. Let $i < m - 1$ and assume $V_{i+1} =$

$U_{i+1}+\sum_{j=i+2}^{m} V_{i+1j}$. For $v \in V_i$ we have $v = v'/w$ where $v' \in V_{i+1}$ and w is the product of N (say) linear forms of T_{i+1}. If $N = 0$ then $v \in V_{i+1} = V_{ii+1}$ and there is nothing to prove. For $N > 0$, by the inductive hypothesis there exists $u \in U_{i+1}$ such that $v' - u \in \sum_{j=i+2}^{m} V_{i+1j}$ and so $v - u/w \in \sum_{j=1+2}^{m} V_{ij}$. Thus it suffices to show that $1/wx_{i+2}^{\rho_{i+2}} \dots x_m^{\rho_m} \in U_i + \sum_{j=i+1}^{m} V_{ij}$, for then $u/w, v \in U_i + \sum_{j=i+1}^{m} V_{ij}$ follow immediately.

If l is a linear form in $x_{i+2}, ..., x_m$, then $l^{\rho}/x_{i+2}^{\rho_{i+2}} ... x_m^{\rho_m} \in \sum_{j=i+2}^{m} V_{i+ij}$ $\subseteq \sum_{j=i+2}^{m} V_{ij}$ for $\rho \geq \sum_{j=i+2}^{m} \rho_j$. So the identity

$$1/(x_{i+1}-l) = \sum_{r=1}^{\rho} l^{r-1}/x_{i+1}^{r} + l^{\rho}/x_{i+1}^{\rho}(x_{i+1}-l)$$

implies that $u'/w(x_{i+1}-l)x_{i+2}^{\rho_{i+2}} ... x_m^{\rho_m} - \sum_{r=1}^{\rho} u'l^{r-1}/wx_{i+1}^{r} x_{i+2}^{\rho_{i+2}} ... x_m^{\rho_m} \in \sum_{j=i+2}^{m} V_{ij}$ for any $w \in T_{i+1}$ and $u' \in R[x_{i+2}, \dots, x_n]$. Now from a simple induction on N the number of linear factors of $w \in T_{i+1}$ we deduce $1/wx_{i+2}^{\rho_{i+2}} ... x_m^{\rho_m} \in U_i + \sum_{j=i+1}^{m} V_{ij}$.

To show this sum is direct we again proceed by induction on $m - i$. The case $m - i = 0$ is trivial as $\sum_{j=i+1}^{m} V_{ij} = 0$ in this case. Suppose $m - i > 0$ and we have a non-zero element u of U_i and $\sum_{j=i+1}^{m} V_{ij}$ written in the form

$$u_1'x_m^{-1} + ... + u_s'x_m^{-s} = \sum_{j=i+1}^{m} P_j w_{i+1}^{-1} ... w_{j-1}^{-1} w_{j+1}^{-1} ... w_m^{-1},$$

where u_k' is in the R-module U_i' generated by power products $x_1^{\rho_1} ... x_{m-1}^{\rho_{m-1}}$ with $\rho_1, \dots, \rho_i \geq 0; \rho_{i+1}, \dots, \rho_{m-1} < 0 (k = 1, \dots, s)$ with $u_s' \neq 0$, $P_j \in R[x]$ for $j = i+1, ..., m$ and $w_j \in T_j$ for $j = i+1, ..., m$. The element w_m of T_m is of the form αx_m^t for some unit α of R. If $t < s$ we can multiply by x_m^s and put $x_m = 0$ to deduce $u_s' = 0$ – a contradiction. If $t > s$ we can multiply by x_m^t and put $x_m = 0$ to deduce $0 = \sum_{j=i+1}^{m-1} \overline{P}_j \overline{w}_{i+1}^{-1} ... \overline{w}_{j-1}^{-1} \overline{w}_{j+1}^{-1} ... \overline{w}_{m-1}^{-1} \alpha^{-1}$, where $\overline{P}_j = P_j(x_1, ..., x_{m-1}, 0)$ and $\overline{w}_j = w_j(x_j, \dots, x_{m-1}, 0) \in T_j (j = i+1, \dots, m-1)$. Then multiplying by x_m^{-t} and subtracting from the left-hand expression for u has the effect of reducing t by at least 1. Therefore without loss of generality we may assume $t = s$, and by the same process deduce

$$u_s' = \sum_{j=i+1}^{m-1} \overline{P}_j \overline{w}_{i+1}^{-1} ... \overline{w}_{j-1}^{-1} \overline{w}_{j+1}^{-1} ... \overline{w}_{m-1}^{-1} \alpha^{-1},$$

Now the inductive bypothesis applied to the case of $m-1$ variables implies $u'_s = 0$ – a contradiction. Hence the sum is direct.

For $i = 1, ..., m$ we also have a further decomposition

Proposition 2.2. $V_{i-1} = U_{i-1} \oplus U_i \oplus \sum_{j=i+1}^m V_{i-1\,j}$ *as R-modules for* $i = 1, \ldots, m$.

Proof. As $V_{i-1\,i} = V_i = U_i \oplus \sum_{j=i+1}^m V_{ij} \subseteq U_i + \sum_{j=i+1}^m V_{i-1\,j}$ by Lemma 2.1, we have again by 2.1

$$V_{i-1} = U_{i-1} \oplus \left(U_i + \sum_{j=i+1}^{m} V_{i-1\,j} \right).$$

The proof that $U_{i-1} \cap \sum_{j=i+1}^m V_{ij} = 0$ given in 2.1 can be slightly modified to show $U_i \cap \sum_{j=i+1}^m V_{i-1\,j} = 0$. Then the proposition follows.

Next we consider a second generating set $X_1, \ldots, X_m$ with $x_i = \sum_{j=i} c_{ij} X_j (i = 1, \ldots, m)$ where $c_{11}, \ldots, c_{mm}$ are all units in R. The notation T_i, V_i, V_{ij}, U_i above relative to $x_1, \ldots, x_m$ is repeated for $X_1, \ldots, X_m$ to give $T'_i, V'_i, V'_{ij}, U'_i$. Clearly $T'_i = T_i, V'_i = V_i$ and $V'_{ij} = V_{ij}$ but we can only be sure $U'_m = R[x] = U_m$. However, as we shall demonstrate U'_i and U_i are isomorphic in a natural way.

Proposition 2.3. *For* $i = 1, \ldots, m$ *and* $v \in V_{i-1}$ *let* $v = f'_{i-1}(v) \oplus f'_i(v) \oplus g'_i(v)$ *be the decomposition of* v *in the direct sum* $V_{i-1} = U_{i-1} \oplus U_i \oplus \sum_{j=i+1}^m V_{i-1\,j}$ *and similarly for* $V_{i-1} = V'_{i-1} = U'_{i-1} \oplus U'_i \oplus \sum_{j=i+1}^m V_{i-1\,j}$, *write* $v = f_{i-1}(v) \oplus f_i(v) \oplus g_i(v)$.

The maps $f_i : U_i \to U'_i$ *and* $f'_i : U'_i \to U_i$ *are inverse isomorphisms* $(i = 0, \ldots, m)$ *with* $f_m = f'_m$ *the identity.*

Remark. If $v \in V_i$ we have $v = f'_i(v) \oplus f'_{i+1}(v) \oplus g'_{i+1}(v)$ with $f'_i(v) \in U_i$ and $f'_{i+1}(v) \oplus g'_{i+1}(v) \in U_{i+1} \oplus \sum_{j=i+2}^m V_{ij} \subseteq \sum_{j=i+1}^m V_{i-1\,j}$. So, as an element of $V_{i-1} \supseteq V_i, v = 0 \oplus f'_i(v) \oplus (f'_{i+1}(v) + g'_{i+1}(v))$. Thus $f'_{i-1}(v) = 0$ and the two meanings for $f'_i(v)$ coincide.

Proof. For $v \in U_i (i = 0, \ldots, m-1)$ we have $v = v \oplus 0 \oplus 0$ as an element of $V_i = U_i \oplus U_{i+1} \oplus \left(\sum_{j=i+2}^m V_{i\,j} \right)$ and $= f_i(v) \oplus f_{i+1}(v) \oplus$

$g_{i+1}(v)$ as an element of $U'_i \oplus U'_{i+1} \oplus \left(\sum_{j=i+2}^{m} V_{ij}\right)$. Then considering $f_i(v), f_{i+1}(v), g_{i+1}(v)$ in turn as elements of $V_i = U_i \oplus U_{i+1} \oplus \left(\sum_{j=i+2}^{m} V_{ij}\right)$ we have $f_i(v) = f'_i f_i(v) \oplus f'_{i+1} f_i(v) \oplus g'_i f_i(v), f_{i+1}(v) = 0 \oplus f'_{i+1} f_{i+1}(v) \oplus g'_i f_{i+1}(v), g_{i+1}(v) = 0 \oplus 0 \oplus g_{i+1}(v)$. Adding we deduce $v = f'_i f_i(v), 0 = f'_{i+1} f_i(v) + f'_{i+1} f_{i+1}(v), 0 = g'_i f_i(v) + g'_i f_{i+1}(v) + g_{i+1}(v)$. Interchanging the rôles of f_i, f'_i we deduce $f_i f'_i : U'_i \to U'_i$ is also the identity and, as f_i, f'_i are both R-homomorphisms, f_i, f'_i are inverse isomorphisms for $i = 0, \ldots, m-1$.

For $v \in U_m$ we have $v = 0 \oplus v \oplus 0$ as an element of both V_{m-1} and V'_{m-1}; i.e. $v = f_m(v) = f'_m(v)$ and f_m, f'_m are both the identity $R[x] \to R[x]$.

Next we consider $U_i (i = 0, \ldots, m)$ as R-submodules of $R[x_1, \ldots, x_m; x_1^{-1}, \ldots, x_m^{-1}] = U$ (say) and U as an $R[x]$- submodule of V_0.

Let $M_1 = \bigoplus_{i=1}^{n} R y_i$ be a free R-module of rank n and $a_1, \ldots, a_n \in R[x]$ be homogeneous polynomials of positive degrees $\rho_1, \ldots \rho_n$. We construct the alternating algebra $M = \bigoplus_{j=0}^{m} M_j$, where M_j is the j-fold alternating product $\bigwedge_j M_1$. Let $d : V_0 \otimes_R M \to V_0 \otimes_R M$ be the $R[x]$-homomorphism with

$$d(v \otimes y_{i_1} \wedge \ldots \wedge y_{i_p}) = \sum_{q=1}^{p} (-1)^{q-1} a_{i_q} v \otimes y_{i_1} \wedge \ldots \hat{y}_{i_q} \ldots \wedge y_{i_p},$$

so $d^2 = 0$ and d restricts to

$$d : U \otimes_R M \to U \otimes_R M, d : V_i \otimes_R M \to V_i \otimes_R M$$

and

$$d : V_{ij} \otimes_R M \to V_{ij} \otimes_R M$$

for $i = 0, \ldots, m$ and $0 \leq i < j \leq m$, all denoted by the same symbol d.

It is clear that for each subset S of $\{1, \ldots, m\}$ we have an R- submodule U_S of U generated by the power products $x_1^{\rho_1} \ldots x_m^{\rho_m}$ with $\rho_i \geq 0$ for $i \in S$ and $\rho_i < 0$ for $i \notin S$, and that U is the direct sum of all the U_S. In particular $U_0, \ldots, U_m$ are direct summands of U and $U_0 \otimes_R M, \ldots, U_m \otimes_R M$ are direct summands of $U \otimes_R M$. For $u \in U \otimes_R M = \oplus U_S \otimes_R M$ we denote by $d_i(u)$ the component of $d(u)$ in $U_i \otimes_R M$. Thus if $u \in U_{i-1} \otimes_R M \subseteq$

$V_{i-1} \otimes_R M (i = 1, \dots, m)$ and V_{i-1} is written as the direct sum of Proposition 2.2, then $d(u) = d_{i-1}(u) \oplus d_i(u) \oplus (d(u) - d_i(u) - d_{i-1}(u))$. In particular $u \in U_m \otimes_R M \subseteq V_{m-1} \otimes_R M$ has $d(u) = 0 \oplus d(u) \oplus 0$, i.e. $d_m = d : U_m \otimes_R M \to U_m \otimes_R M$.

We return to the generating set $X_1, \dots, X_m$ of $R[x]$ with $x_i = \sum_{j=i}^m c_{ij} X_j (i = 1, \dots, m)$ where $c_{11}, \dots, c_{mm}$ are all units of R. Consider U', U'_i, d'_i defined relative to $X_1, \dots, X_m$; we abuse the notation of Proposition 2.3 so that $f_i : U_i \otimes_R M \to U'_i \otimes_R M$ is short for $f_i \otimes M$.

Proposition 2.4.

(i) If $u \in U_i \otimes_R M$, then $f_i d_i(u) = d'_i f_i(u)$ for $i = 0, \dots, m$ and $f_{i+1} d_{i+1}(u) + f_{i+1} d_i(u) = d'_{i+1} f_{i+1}(u) + d'_{i+1} f_i(u)$ for $i = 0, \dots, m-1$.

(ii) If $u \in U_0 \otimes_R M$ and $d_0(u) = 0$, then

$$f_m d_m \dots d_1(u) = d'_m \dots d'_1 f_0(u) + \sum_{t=1}^{m} d'_m \dots d'_t f_t d_{t-1} \dots d_1(u)$$

Proof. (i) Let $u \in U_i \otimes_R M \subseteq V_i \otimes_R M (i = 0, \dots, m-1)$. Express $d(u)$ first as a sum of components in $(U_i \otimes_R M) \oplus (U_{i+1} \otimes_R M) \oplus \left(\left(\sum_{j=i+2}^m V_{ij}\right) \otimes_R M\right)$ and then those components in turn as elements of $(U'_i \otimes_R M) \oplus (U'_{i+1} \otimes_R M) \oplus \left(\left(\sum_{j=i+2}^m V_{ij}\right) \otimes_R M\right)$. The first two components of the resulting expression for $d(u)$ are $f_i d_i(u)$ and $f_{i+1} d_i(u) + f_{i+1} d_{i+1}(u)$. On the other hand $u = f_i(u) \oplus f_{i+1}(u) \oplus g_{i+1}(u) \in (U'_i \otimes_R M) \oplus (U'_{i+1} \otimes_R M) \oplus \left(\left(\sum_{j=i+2}^m V_{ij}\right) \otimes_R M\right)$ and operating with d' gives the first two components $d'_i f_i(u)$ and $d'_{i+1} f_i(u) + d'_{i+1} f_{i+1}(u)$. So (i) is proved for $i = 0, \dots, m-1$. For $i = m$ we have noted above $d_m = d = d'_m$ and f_m is identity so $f_m d_m(u) = d'_m f_m(u)$ is trivial.

(ii) Let $u \in U_0 \otimes_R M$ satisfy $d_0(u) = 0$; we prove $f_r d_r \dots d_1(u) = d'_r \dots d'_1 f_0(u) + \sum_{t=1}^r d'_r \dots d'_t f_t d_{t-1} \dots d_1(u)$ for $r = 0, \dots, m$ by induction on r. The case $r = 0$ is trivial and $r = 1$ follows from the second equality of (i) when $i = 0$. Assume $m \geq k > 1$ and the required equality holds for

$r = k - 1$. Applying d'_k we have

$$d'_k f_{k-1} d_{k-1} ... d_1(u) = d'_k ... d'_1 f_0(u) + \sum_{t=1}^{k-1} d'_k ... d'_t f_t d_{t-1} ... d_1(u).$$

But by (i) $d'_k f_{k-1} = (f_k d_k + f_k d_{k-1} - d'_k f_k) : U_{k-1} \otimes M \to U'_{k-1} \otimes_R M$, so the inductive step follows by showing $f_k d_{k-1} d_{k-1} ... d_1(u) = 0$. The component of $0 = d^2 : V_i \otimes_R M \to V_i \otimes_R M$ which maps $U_i \otimes_R M \to U_{i+1} \otimes_R M$ is $d_{i+1} d_i + d_{i+1} d_{i+1} = 0$, therefore $d_{k-1} d_{k-1} \ldots d_1(u) = (-1)^{k-1} d_{k-1} ... d_1 d_0(u) = 0$ as required.

The grading of $R[x]$ in which each indeterminate x_i has degree 1 extends naturally to V_0 by assigning to a quotient P/Q, with P homogeneous, the degree

$$\deg(P/Q) = \deg(P) - \deg(Q),$$

so all the $R[x]$-modules V_i, V_{ij} and R-modules are also graded. The isomorphisms f_i, f'_i are graded (of degree zero), and, if we assign to the generator y_j of M the same degree ρ_j as the homogeneous polynomial $a_j (j = 1, ..., n)$, then the maps d and $d_i (i = 0, ..., m)$ are graded.

In the following theorem we use the notation of Proposition 2.4. From [3; Theorem 3.3] recall that in terms of $x_1, ..., x_m$, *core* $(a_1, ..., a_n)$ is the ideal of elements of $R \cong R \otimes_R M_0$ which are of the form $d_m ... d_1 u$, where u is of degree 0 in $U_0 \otimes_R M_m$ with $d_0(u) = 0$. So the theorem is equivalent to the statement that if the generators $x_1, ..., x_m$ and $X_1, ..., X_m$ of $R[x]$ are related by $x_i = \sum_{j=i}^{m} c_{ij} X_j (i = 1, ..., m)$ with $c_{11}, ..., c_{mm}$ all units of R then *core* $(a_1, ..., a_n)$ defined in terms of $x_1, ..., x_m$ is the same as that defined in terms of $X_1, ..., X_m$.

Theorem 2.5. *With the notation of 2.3 and 2.4 let $u \in U_0 \otimes_R M_m$ have degree 0, then*

(i) $f_0(u) \in U'_0 \otimes_R M_m$ has degree 0 and $d_0(u) = 0$ if and only if $d'_0 f_0(u) = 0$; and

(ii) if $d_0(u) = 0$ then

$$d_m ... d_1(u) = d'_m ... d'_1 f_0(u).$$

Proof. (i) As noted in the preamble the isomorphisms f_i, f'_i are graded so $f_0(u)$ has degree 0. By 2.4 (i) $f_0 d_0(u) = d'_0 f_0(u)$, so $d_0(u) = 0$ implies $d'_0 f_0(u) = 0$ and the converse holds because f_0 is an isomorphism (2.3).

Also by 2.4 (ii) $d_0(u) = 0$ implies

$$f_m d_m ... d_1(u) = d'_m ... d'_1 f_0(u) + \sum_{t=1}^{m} d'_m ... d'_t f_t d_{t-1} ... d_1(u)$$

On the left we note that f_m is the identity (2.3). As for the right-hand sum, first $d'_m ... d'_{t+1} f_t d_{t-1} ... d_1(u) \in U'_m \otimes_R M_1 \simeq \bigoplus_{i=1}^{n} R[x] y_i$ with degree 0, therefore $d'_m ... d'_{t+1} f_t d_{t-1} ... d_1(u) = 0 (t = 1, \ldots, m)$. But, as we noted at the end of the proof of 2.4, $0 = d^2 : V_i \otimes_R M \to V_i \otimes_R M$, so the component $d'_{i+1} d'_i + d'_{i+1} d'_{i+1} : U'_i \otimes_R M \to U'_{i+1} \otimes_R M$ is also zero $(i = 0, ..., m-1)$. Hence $0 = d'_m d'_m ... d'_{t+1} f_t d_{t-1} ... d_1(u) = (-1)^{m-t} d'_m ... d'_{t+1} d'_t f_t d_{t-1} ... d_1(u)$ and $d_m ... d_1(u) = d'_m ... d'_1 f_0(u)$ follows.

3. THE GENERAL CASE

We shall use the symbols $V_i, V_{ij}, U_i, U, d_i, d$ as in the last section relative to the generating set $x_1, ..., x_m$ of the R-algebra $R[x]$. But now the same symbols primed will be relative to other generating sets $X_1, ..., X_m$, not only those related to $x_1, ..., x_m$ via an upper triangular matrix. Our first proposition is taken from Kirby [3; Theorem 3.1]. The reader should beware that the symbols U_i, U'_i, d_i, d'_i in [3] have a meaning slightly different from that of the present note, although they come together in the proposition.

Proposition 3.1. *Let $X_1 ..., X_m$ be a permutation of $x_1, ..., x_m$ with signature σ. If $u \in U_0 \otimes_R M_m$ has degree zero and $d_0(u) = 0$, then $d'_0(u) = 0$ and*

$$d_m ... d_1(u) = \sigma d'_m ... d'_1(u).$$

Recall, for homogeneous polynomials $a_1, ..., a_n$ of positive degrees in $R[x]$, the definition of the ideal *core* $(a_1, ..., a_n)$ in R is heavily dependent on the ordered generating set $(x_1, ..., x_m)$ of $R[x]$. Proposition 3.1 implies that *core* $(a_1, ..., a_n)$ is independent of the order of $x_1, ..., x_m$, and Theorem 2.5 implies *core* $(a_1, ..., a_n)$ is unchanged when $x_1, ..., x_m$ is replaced by a second m-element generating set related to $x_1, ..., x_m$ by an upper triangular invertible matrix. The general case for R a quasi-local ring results by combining 2.5 and 3.1.

Lemma 3.2. *When R is a quasi-local ring core $(a_1, ..., a_n)$ is independent of the m-element generating set of the R-algebra $R[x_1, ..., x_m]$.*

Proof. Let A be an $m \times m$ invertible matrix over R, so $\det A \notin M$, the maximal ideal of R. Each row (and column) of A contains an entry which is a unit of R. So the standard Gauss elimination process shows that there exist an invertible lower triangular matrix L, an invertible upper triangular matrix U and a permutation matrix P such that $A = LUP$. If Q denotes the $m \times m$ permutation matrix that reverses order then $A = Q(QLQ)QUP$ is a decomposition of A as a product of permutation and upper triangular matrices.

Now if $X_1, ..., X_m$, is a second generating set of $R[x_1, ..., x_m]$ we have $x_1 ... x_m)^t = A(X_1 ... X_m)^t$ for some invertible matrix A over R. Therefore the lemma follows from three applications of 3.1 and two applications of 2.5.

Our main result now follows by localization.

Theorem 3.3. *$core(a_1, ..., a_n)$ is independent of the m-element generating set $x_1, ..., x_m$ of $R[x_1, ..., x_m]$.*

Proof. For each maximal ideal P of R let g_P denote the natural ring homomorphism $R \to R_P$. We use the same symbol also to denote the R-homomorphism $R \otimes_R N \to R_P \otimes_R N$ for all R-modules N. In view of 3.2 and with *core* defined in terms of the generating set $x_1, \ldots, x_m$ of $R[x_1, \ldots, x_m]$ and $R_P[x_1, \ldots, x_m]$ for all maximal ideals P we must prove, for $\alpha \in R, \alpha \in$ *core* $(a_1, \ldots, a_n)$ if and only if $g_P(\alpha) \in$ *core* $(g_P(a_1), \ldots g_P(a_n))$ for all maximal ideals P.

For each P and $i = 0, \ldots, m$ let $d_P = R_P \otimes d, (d_P)_i = R_P \otimes d_i$ be the localizations of the boundary homomorphisms d and d_i associated with

$a_1, \ldots, a_n$; so $d_P, (d_P)_i$ are similarly associated with $g_P(a_1), \ldots, g_P(a_1)$. The commutative properties $g_P d_i = (d_P)_i g_P$ are immediate, and it is clear that g_P preserves degrees.

Let $u \in U_0 \otimes_R M_m$ have degree 0 with $d_0(u) = 0$, so $d_m \ldots d_1(u)$ is a typical element of *core* $(a_1, \ldots a_n)$. Then, for all maximal $P, g_P(u)$ has degree zero in $R_P \otimes_R U_0 \otimes_R M_m$, $(d_P)_0(g_P(u)) = g_P d_0(u) = 0$ and $g_P d_m \ldots d_1(u) = (d_P)_m \ldots (d_P)_1(g_P(u)) \in$ *core* $(g_P(a_1), \ldots, g_P(a_n))$.

Conversely suppose $\alpha \in R$ has $g_P(\alpha) \in$ *core* $(g_P(a_1, \ldots, g_P(a_n))$ for all maximal ideals P. Then, for each such P, there exists $u_P \in R_P \otimes_R U_0 \otimes_R M_m$ of degree zero such that $(d_P)_0(u_P) = 0$ and $g_P(\alpha) = (d_P)_m \ldots (d_P)_1(u_P)$. So, for some $\sigma_P \in R \backslash P$, there exists $u'_P \in U_0 \otimes_R M_m$ of degree zero such that $g_P(u'_P) = \sigma_P u_P$, then $g_P d_0(u'_P) = (d_P)_0(\sigma_P u_P) = 0$ and $d_0(\sigma'_P u'_P) = 0$ for some $\sigma'_P \in R \backslash P$. Hence $g_P(\sigma'_P \sigma_P \alpha) = (d_P)_m \ldots (d_P)_1 g_P (\sigma'_P u'_P) = g_P d_m \ldots d_1(\sigma'_P u'_P)$, and, for some $\sigma''_P \in R \backslash P$, $\sigma''_P \sigma'_P \sigma_P \alpha = d_m \ldots d_1(\sigma''_P \sigma'_P u'_P) \in$ *core* $(a_1, \ldots, a_n)$. Therefore, for each maximal ideal P, *core* $(a_1, \ldots, a_n) :_R R\alpha \not\subseteq P$ which implies $\alpha \in$ *core* $(a_1, \ldots, a_n)$ as required.

4. CHANGE OF GENERATORS $a_1, \ldots, a_n$

In this section we consider a second set of forms $b_1, \ldots, b_p$ of $R[x]$ with positive degrees such that $b_i = \sum_{j=1}^{n} \beta_{ij} a_j (i = 1, \ldots, p)$ with β_{ij} forms of $R[x]$. Let $M' = \bigoplus_{i=0}^{p} M'_i$ be the alternating algebra on the free module $M'_1 = \sum_{i=1}^{p} R z_i$ and $d' : U \otimes_R M' \to U \otimes_R M'$ the boundary homomorphism of the Koszul complex $K(b_1, \ldots, b_p; U)$ and derive $d'_i (i = 0, \ldots, m)$ from d' as d_i is derived from d the boundary homomorphism of $K(a_1, \ldots, a_n; U)$ in §2.

In the proof of 2.4 (ii) we noted that $d^2 = 0$ implies

$$0 = d_{i+1} d_i + d_{i+1} d_{i+1} : U_i \otimes_R M \to U_{i+1} \otimes_R M \quad (i = 0, \ldots, m-1)$$

so also

$$0 = d'_{i+1} d'_i + d'_{i+1} d'_{i+1} : U_i \otimes_R M \to U_{i+1} \otimes_R M' \quad (i = 0, \ldots, m-1).$$

Further the mapping $g : U \otimes_R M' \to U \otimes_R M$ obtained by extending the graded $R[x]$-homomorphism $U \otimes_R M_1' \to U \otimes_R M_1$ with $u \otimes z_i \to \sum_{j=1}^n \beta_{ij} u \otimes y_j$ $(i = 1, \dots, p)$ is clearly a morphism of complexes. We denote by g_i the component of g with image in the direct summand $U_i \otimes_R M$ of $U \otimes_R M$. Then $gd' = dg$ implies

$$g_i d_i' = d_i g_i : U_i \otimes_R M' \to U_i \otimes_R M \quad (i = 0, \dots, m)$$

and $g_{i+1} d_i' + g_{i+1} d_{i+1}' = d_{i+1} g_i + d_{i+1} g_{i+1} : U_i \otimes_R M' \to U_{i+1} \otimes_R M$.

With these basic properties of d_i, d_i' and $(i = 0, \dots, m)$ the proofs of 2.4 (ii) and 2.5 can be adapted immediately to establish the following two companion results.

Proposition 4.1. *If $u \in U_0 \otimes_R M'$ and $d_0' u = 0$, then for $r = 0, \dots, m$*

$$g_r d_r' \dots d_1'(u) = d_r \dots d_1 g_0(u) + \sum_{t=1}^{r} d_r \dots d_t g_t d_{t-1} \dots d_1(u).$$

Theorem 4.2. *If $u \in U_0 \otimes_R M_m'$ has degree 0 and $d_0' u = 0$, then (i) $g_0(u) \in U_0 \otimes_R M_m$ has degree 0 with $d_0 g_0(u) = 0$, and (ii) $g_m d_m' \dots d_1'(u) = d_m \dots d_1 g_0(u)$.*

$$g_m d_m' \dots d_1'(u) = d_m' \dots d_1'(u) \in R \otimes_R R \cong R,$$

and the following corollary is immediate.

Corollary 4.3. *If $\sum_{i=1}^n a_i R[x] \supseteq \sum_{j=1}^p b_j R[x]$, then core $(a_1, \dots, a_n) \supseteq$ core $(b_1, \dots, b_p)$. In particular, if $\sum_{i=1}^n a_i R[x] = \sum_{j=1}^p b_j R[x]$, then core $(a_1, \dots, a_n) =$ core $(b_1, \dots, b_p)$.*

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