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FUZZY DERIVATIONS

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ABSTRACT. We define a fuzzy derivation and prove some of its basic properties. We show not only that a fuzzy derivation is a fuzzy linear transformation, but also that the Lie commutator of two fuzzy derivations is a fuzzy derivation. We show that a fuzzy derivation can be determined by a fuzzy isomorphism of the algebra of dual numbers. We determine the set of D-constants of a fuzzy derivation D. We also study properties of fuzzy derivations in their own right, i.e., properties which are inherently fuzzy and thus not modifications of crisp results.

INTRODUCTION

Zadeh [27] followed by Rosenfeld [24] inspired the study of fuzzy abstract algebra. In this paper, we continue the study by introducing the concept of fuzzy derivations. Derivations have been used extensively in modern Galois theory, [2,3,4,7,11,12,13,14,17]. A more extensive reference list can be found in [4]. Classical Galois theory involving fuzzy subgroups and fuzzy subfields has been introduced in [18,19]. Hence the time is ripe for the introduction of fuzzy derivations in order to begin the study of modern Galois theory dealing with certain fuzzy algebraic substructures. Also, derivations are closely connected to power series, [8,9,10], and consequently have applications in computer science, [8]. Applications of fuzzy power series to computer science was started in [23]. Fuzzy power series subrings have been examined in [5,6].

Let A and B be associative and commutative algebras over a field K. A derivation d of A into B is a linear transformation of A into B such that $\forall x, y \in A$, $d(xy) = d(x)y + xd(y)$. Hence it is natural to define a fuzzy derivation of A into B as a particular type of fuzzy function. There are several definitions of fuzzy function to choose from. We choose the one which is closest to the notion of a crisp function. Our reason for doing so is because it is essential that the composition of two fuzzy derivations be a fuzzy linear operator so that the Lie commutator of two fuzzy derivations is a fuzzy derivation. It is known that the composition of two fuzzy functions is not in general a fuzzy function for any definition of fuzzy function of current interest other than the one we select, [21]. Our purpose is to introduce the basic ideas of a fuzzy derivation. The complications caused by not having the composition of two fuzzy functions a fuzzy function can be more easily examined once the basic ideas are put forth. A reason for requiring the composition of two fuzzy functions to be a fuzzy function is that we will eventually need the set of all fuzzy derivations from A into B to be a type of Lie algebra. The notion of fuzzy Lie subalgebras has been introduced in [26].

In Section 1, we lay down the necessary preliminary results. We also show here a difference which arises from our definition of a fuzzy function in comparison to that of crisp function, Example 1.11. In Section 2, we define a fuzzy derivation and prove some of its basic properties. We show not only that a fuzzy derivation is a fuzzy linear transformation, but also that the Lie commutator of two fuzzy derivations is a fuzzy derivation, Theorem 2.2. We show that a fuzzy derivation can be determined by a fuzzy isomorphism of the *algebra of dual numbers* $B \otimes_K T$, [14], where $T = K[x]/ < x^2 >$ and x is an indeterminant over K, Theorem 2.3. In Section 3, we determine the set of D-constants of a fuzzy derivation D, Proposition 3.2. We also study properties of fuzzy derivations in their own right, i.e., properties which are inherently fuzzy and thus not modifications of crisp results, Theorems 3.7 and 3.10.

We recall that fuzzy subset of a set X is a function of X into the closed interval [0,1]. Let X be a set and f a fuzzy subset of X. Let $t \in [0,1]$ and $f_t = \{x \in X | f(x) \geq t\}$. Then f_t is called a *level set.* Let $f^* = \{x \in X | f(x) > 0\}$. Then f^* is called the *support* of f. f is a fuzzy subgroup of a group G if f is fuzzy subset of G such that $f(xy^{-1}) \geq \min\{f(x), f(y)\} \; \forall x, y \in G$.

1. PRELIMINARY RESULTS

In the following, we let A and B be associative and commutative algebras over the field K. Consider the following conditions on a fuzzy subset T of the Cartesian-cross product $A \times B$:

(1) $\forall x \in A, \exists y \in B$ such that $T(x, y) > 0$,

(2) $\forall y \in B, \exists x \in A$ such that $T(x, y) > 0$,

(3) $\forall x \in A, \forall y_1, y_2 \in B, T(x, y_1) > 0$ and $T(x, y_2) > 0$ implies $y_1 = y_2$,

(4) $\forall x_1, x_2 \in A, \forall y \in B, T(x_1, y) > 0$ and $T(x_2, y) > 0$ implies $x_1 = x_2$.

The following definition of fuzzy function has appeared naturally in the study of fuzzy graphs and their application to cluster analysis [25] and in fuzzy integration [20].

Definition 1.1. Let T be a fuzzy subset of $A \times B$. If condition (3) holds, then T is called a *fuzzy function* of A into B. If conditions (3) and (4) hold, then T is called a one-to-one fuzzy function of A into B.

Definition 1.2. Let T be a fuzzy subset of $A \times B$. Define the *domain* of T, written $\mathcal{D}(T)$, to be $\{x \in A | \exists y \in B, T(x, y) > 0\}$. Define the *image* of T, written $\mathcal{I}(T)$, to be $\{y \in B | \exists x \in A, T(x, y) > 0\}$. If $\mathcal{I}(T) = B$, then T is said to map $\mathcal{D}(T)$ onto B.

If T is a fuzzy function of A onto A such that $T(x, x) > 0$ $\forall x \in A$, then T is called an *identity* on A.

Definition 1.3. Let T be a fuzzy function of A into B. Then T is called a fuzzy linear operator of A into B if and only if $\forall x_1, x_2 \in A, \forall y \in B, \forall c \in K$,

(i) $T(x_1 + x_2, y) \geq \sup\{\min\{T(x_1, y_1), T(x_2, y_2)\} | y = y_1 + y_2\}$,

(ii) $T(cx_1, cy) \geq T(x_1, y)$.

We let $\mathcal{FL}(A,B)$ denote the set of all fuzzy linear operators of A into B.

If $T \in \mathcal{FL}(A,B)$, then we let $T^* = \{x \in A | T(x,0) > 0\}$.

Proposition 1.4. [1] *Let $T \in \mathcal{FL}(A,B)$. Then*

(i) $\mathcal{D}(T)$ and $\mathcal{I}(T)$ are subspaces of A and B, respectively,

(ii) $T(0,0) > 0$,

(iii) $\forall x \in A, \forall y \in B, T(x,y) > 0 \Rightarrow T(-x,-y) > 0$,

(iv) T^* is a subspace of A,

(v) T is one-to-one if and only if $T^* = \{0\}$.

Throughout the remainder of the paper, we assume that $\mathcal{D}(T) = A$.

Definition 1.5. Define $+$ on $\mathcal{FL}(A,B)$ by $\forall S,T \in \mathcal{FL}(A,B)$, $(S+T)(x,y) = \sup\{\min\{S(x,y_1),T(x,y_2)\} \mid y = y_1 + y_2,\ y_1,y_2 \in B\}$ $\forall (x,y) \in A \times B$. Define $\cdot$ between K and $\mathcal{FL}(A,B)$ by $\forall c \in K, c \neq 0$, and $\forall D \in \mathcal{FL}(A,B), (cD)(x,y) = D(x,c^{-1}y)$ and $(0D)(x,y) = 0$ if $y \neq 0$ and $(0D)(x,y) = \sup\{D(x,z) | z \in B\}$ if $y = 0$ $\forall (x,y) \in A \times B$.

We note that $\forall x \in A, \exists$ unique $z \in B$ such that $D(x,z) > 0$ and so $(0D)(x,0) = D(x,z)$.

Proposition 1.6. [1] *Let $S,T \in \mathcal{FL}(A,B)$ and let $c \in K$. Then*

(i) $(S+T)(x,y) = \min\{S(x,y_1),T(x,y_2)\}$ if $y = y_1+y_2$, $S(x,y_1) > 0$, and $T(x,y_2) > 0$; otherwise $(S+T)(x,y) = 0$.

(ii) $(cS)(x,cy) = S(x,y)$ if $c \neq 0$.

For $t \in [0,1]$, let $\mathcal{FL}_t(A,B) = \{D \in \mathcal{FL}(A,B) | D(A \times B) = \{0,t\}\}$. Let $\mathcal{FL}^*(A,B) = \cup_{t \in [0,1]} \mathcal{FL}_t(A,B)$.

Theorem 1.7. [1] *(i)* $(\mathcal{FL}(A,B),+)$ *is a commutative semigroup with identity.*

(ii) $(\mathcal{FL}_t(A,B),+)$ *is a commutative group, where* $t \in [0,1]$.

(iii) $(\mathcal{FL}^*(A,B),+)$ *is a completely regular semigroup.*

Theorem 1.8. [1] $\mathcal{FL}(A,B), \mathcal{FL}_t(A,B)$, *and* $\mathcal{FL}^*(A,B)$ *are closed under multiplication by elements of* K. *Furthermore,* $\forall\ S,T \in \mathcal{FL}(A,B)$ *and* $\forall\ c,d \in K$,

(ii) $(cd)S = c(dS)$,

(iii) $c(S+T) = cS + cT$,

(iv) $(c+d)S = cS + dS$,

(iv) $1S = S$.

It is clear that for $t \in (0,1], \mathcal{FL}_t(A,B) \cong \mathcal{L}(A,B)$, where $\mathcal{L}(A,B)$ is the vector space of all linear transformations of A into B.

The following result is well known and so we omit its proof.

Proposition 1.9. *Let* $(G,*)$ *and* $(H,*')$ *be groups. Then the following statements are equivalent.*

(1) f *is a homomorphism of* G *into* H.

(2) f *is a subgroup of* $(G \times H, \otimes)$ *and* f *is a function of* G *into* H.

Let $(G,*)$ and $(H,*')$ be groups. Then f is a *fuzzy homomorphism* of G into H if f is a fuzzy function of G into H such that $f(x_1 * x_2, y) \geq \sup\{\min\{f(x_1,y_1), f(x_2,y_2)\} | y = y_1 *' y_2, y_1, y_2 \in H\}$.

Proposition 1.10. *Let* $(G,*)$ *and* $(H,*')$ *be groups. Then statement (2) implies statement (1). If* G *is finite, then statement (1) implies statement (2).*

(1) f *is a fuzzy homomorphism of* G *into* H.

(2) f is a fuzzy subgroup of $(G \times H, \otimes)$ and f is a fuzzy function of G into H.

Proof. (1) $\Rightarrow$ (2): $f((x,y) \otimes (u,v)) = f(x * u, y *' v) \geq \sup\{\min\{f(x,z), f(u,w)\} | y *' v = z *' w,\ z, w \in H\} \geq \min\{f(x,y), f(u,v)\}$. Hence f is a fuzzy subsemigroup of $G \times H$. Hence the level sets of f are subsemigroups of $G \times H$ and since G is finite, the level sets of f must be subgroups. Thus f is a fuzzy subgroup of $G \times H$.

(2) $\Rightarrow$ (1): $f(x * u, y *' v) = f((x,y) \otimes (u,v)) \geq \min\{f(x,y), f(u,v)\}$ and $\sup\{\min\{f(x,z), f(u,w)\} | y *' v = z *' w, z, w \in H\} = \min\{f(x,y), f(u,v)\}$ if $f(x,y) > 0, f(u,v) > 0$. Thus if $f(x,y) > 0$ and $f(u,v) > 0$, then $f(x * u, y *' v) \geq \sup\{\min\{f(x,z), f(u,w)\} | y *' v = z *' w, z, w \in H\}$. If either $f(x,y) = 0$ or $f(u,v) = 0$, then the desired equality holds.

In the next example, we show that the statement (1) does not imply statement (2) in general.

Example 1.11. Let $G = H =< z >$ be an (additive) infinite cyclic group. Define $f :< z > \times < z > \longrightarrow [0,1]$ as follows:

$f(0,0) = 1, f(nz, nz) = 3/4$ if $n > 0$, $f(nz, nz) = 1/2$ if $n < 0$, and $f(nz, mz) = 0$ if $n \neq m$. We show that f is a fuzzy homomorphism of $< z >$ into $< z >$, f is a fuzzy subsemigroup of $< z > \times < z >$, but f is not a fuzzy subgroup of $< z > \times < z >$.

Clearly f is a fuzzy function of $< z >$ into $< z >$. If $y = nz + mz$, then $\sup\{\min\{f(nz, y_1), f(mz, y_2) | y = y_1 + y_2, y_1, y_2 \in H\} = \min\{f(nz, nz), f(mz, mz)\} \leq f(nz + mz, y)$. (Note $\min\{f(z,z), f(-z,-z)\} = 1/2 < 1 = f(z - z, 0)$.) If $y \neq nz + mz$, then $\sup\{min\{f(nz, y_1), f(mz, y_2)\} | y = y_1 + y_2, y_1, y_2 \in H\} = 0 = f(nz + mz, y)$. Thus f is an identity and a fuzzy isomorphism of $< z >$ onto $< z >$.

$$= \begin{cases} f((nz, nz) + (mz, mz)) = f(((n+m)z, (n+m)z)) \\ 1 \quad \text{if } m = -n \\ 3/4 \quad \text{if } m > -n \\ 1/2 \quad \text{if } m < -n. \end{cases}$$

$\min\{f(nz,nz),f(mz,mz)\}$

$$= \begin{cases} 1 & \text{if } m=n=0 \\ 3/4 & \text{if } m>-n \text{ and } m,n>0 \\ 1/2 & \text{if } m>-n \text{ and either } m<0 \text{ or } n<0 \\ 1/2 & \text{if } m<-n \text{ since either } m<0 \text{ or } n<0. \end{cases}$$

Also $f((nz,kz)+(mz,lz)) \geq 0 = \min\{f(nz,kz),f(mz,lz)\}$ if either $n \neq k$ or $m \neq l$. Hence $f((nz,kz)+(mz,lz)) \geq \min\{f(nz,kz),f(mz,lz)\}\forall$ $(nz,kz),(mz,lm) \in <z> \times <z>$. Thus f is a fuzzy subsemigroup of $<z> \times <z>$. Now $f(nz,nz) \neq f(-nz,-nz)$ if $n \neq 0$. Thus f is not a fuzzy subgroup of $<z> \times <z>$.

Proposition 1.12. *Let U and V be vector spaces over K. Then T is a linear operator of U into V if and only if T is a subspace of $U \times V$ and T is a function.*

Proposition 1.13. *Let U and V be vector spaces over K. Then T is a fuzzy linear operator of U into V if and only if T is a fuzzy subgroup of $U \times V$ and T is a fuzzy function of U into V.*

Proof. Suppose that T is a fuzzy linear operator of A into B. Then T is a fuzzy function U into V by definition. Let $(x,y),(u,v) \in U \times V$. Then $T((x,y) \oplus (u,v)) = T(x+u,y+v) \geq \sup\{\min\{T(x,w),T(u,z)\}|y+v=w+z\} \geq \min\{T(x,y),T(u,v)\}$. Let $k \in K$. Then $T(kx,ky) = T(k(x,y)) \geq T(x,y)$. Thus $T(-(x,y)) = T(-1(x,y)) = T(-1x,-1y) \geq T(x,y)$. Thus T is a fuzzy subgroup of $U \times V$. Conversely suppose that T is a fuzzy subgroup of $U \times V$ and T is a fuzzy function of U into V. Then $T(x+x',y+y') = T((x,y)\oplus(x',y')) \geq \min\{T(x,y),T(x',y')\}$ and $\sup\{\min\{T(x,z),T(x',z')\}\,|y+y'=z+z',z,z' \in V\} = \min\{T(x,y),T(x',y')\}$ if $T(x,y)>0, T(x',y')>0$. Thus if $T(x,y)>0$ and $T(x',y')>0$, then $T(x+x',y+y') \geq \sup\{\min\{T(x,z),T(x',z')\}|y+y'=z+z',z,z' \in V\}$. If either $T(x,y)=0$ or $T(x',y')=0$, then the desired equality holds. Let $c \in K$. Then $T(cx,xy) = T(c(x,y)) \geq T(x,y)$. Thus T is a fuzzy linear operator of U into V.

Proposition 1.14. *Let C be an associative and commutative algebra over K. Let $T \in \mathcal{FL}(A,B)$ and $S \in \mathcal{FL}(B,C)$. Define the composition* *of T with S, written $S \circ T$, as follows:* $\forall x \in A, \forall z \in C$,

$$(S \circ T)(x,z) = \sup\{\min\{T(x,y), S(y,z)|y \in B\}.$$

Then $S \circ T \in \mathcal{FL}(A,C)$.

Proof. By [16], it follows that $(S \circ T)(x_1 + x_2, z) \geq \sup\{\min\{(S \circ T)(x_1,z_1),(S \circ T)(x_2,z_2)\}|z = z_1 + z_2, z_1, z_2 \in C\}\ \forall x_1, x_2 \in A$. Let $k \in K, k \neq 0$. Then $(S \circ T)(kx, kz) = \sup\{\min\{T(kx,y),\ S(y,kz)|y \in B\} = \sup\{\min\{T(x,k^{-1}y), S(k^{-1}y,z)|y \in B\} = \sup\{\min\{T(x,y'), S(y',z)|y' \in B\} = (S \circ T)(x,z)$. Let $k = 0$. By Definition 1.3, $T(0,0) = T(0x,0y) \geq T(x,y) \forall x \in A, \forall y \in B$ and similarly, $S(0,0) \geq S(y,z)\ \forall y \in B, \forall z \in C$. Thus $(S \circ T)(0x,0z) = (S \circ T)(0,0) = \sup\{\min\{T(0,y),S(y,0)\}|y \in B\} = \min\{T(0,0),S(0,0)\} \geq \min\{T(x,y),S(y,z)\} \forall y \in B$. Hence $(S \circ T)(0x,0z) \geq \sup\{\min\{T(x,y),S(y,z)\}|y \in B\} = (S \circ T)(x,z)$. Thus $S \circ T \in \mathcal{FL}(A,C)$.

Remark 1.15. Let $T \in \mathcal{FL}(A,B)$ and $S \in \mathcal{FL}(B,C)$. Then we note that $(S \circ T)(x,z) = \min\{T(x,y), S(y,z)\}$ if $\exists y \in B$ such that $T(x,y) > 0$ and $S(y,z) > 0$ since such a y is unique if it exists and $(S \circ T)(x,z) = 0$ otherwise. Let $T \in \mathcal{FL}(A,B)$ and $S \in \mathcal{FL}(A,B)$. Then $(S+T)(x,y) = \min\{S(x,y_1), T(x,y_2)\}$ if $\exists y_1, y_2 \in B$ such that $S(x,y_1) > 0$, $T(x,y_2) > 0$. and $y = y_1 + y_2$; and $(S+T)(x,y) = 0$ otherwise by Proposition 1.6.

2. FUZZY DERIVATIONS

In the remainder of the paper, we assume that $K \subseteq A$ and that A is a subalgebra of B.

Definition 2.1. Let $D \in \mathcal{FL}(A,B)$. Then D is called a fuzzy derivation if $\forall x_1, x_2 \in A, \forall y \in B$, $D(x_1x_2, y) \geq \sup\{\min\{D(x_1,y_1), D(x_2,y_2)\}|y = x_2y_1 + x_1y_2, y_1, y_2 \in B\}$. Let $FD(A,B) = \{D \in \mathcal{FL}(A,B)|D$ is a fuzzy derivation $\}$.

It is easy to verify that $\mathcal{FD}(A,B)$ is a "subspace" of $\mathcal{FL}(A,B)$ over K, i.e., $\mathcal{FD}(A,B)$ is closed under addition and scalar multiplication by elements from K. Of course, $\mathcal{FD}(A,B)$ inherits the properties of Theorem 1.7 (i) and Theorem 1.8.

For $D, D' \in \mathcal{FD}(A, A)$, we let DD' denote $D \circ D'$. By $DD' - D'D$, we mean the fuzzy subset $DD' + (-(D'D))$ of $A \times A$. Then $(DD' - D'D)(x, y) = \min\{DD'(x, y_1), D'D(x, y_2)\}$ if $y = y_1 - y_2$ and $DD'(x, y_1) > 0$ and $D'D(x, y_2) > 0$ for some $y_1, y_2 \in A$; otherwise $(DD' - D'D)(x, y) = 0$.

Theorem 2.2. *Let $D, D' \in \mathcal{FD}(A, A)$. Then $DD' - D'D \in \mathcal{FD}(A, A)$.*

Proof. By Proposition 1.14, we know that $DD', D'D \in \mathcal{FL}(A, A)$. By Theorems 1.7 and 1.8, we know that $DD' - D'D \in \mathcal{FL}(A, A)$. Hence it suffices to show that $\forall x_1, x_2, y \in A$, $(DD' - D'D)(x_1x_2, y) \geq \sup\{\min\{(DD' - D'D)(x_1, y_1), (DD' - D'D)(x_2, y_2)\} | y = x_2y_1 + x_1y_2,\ y_1, y_2 \in A\}$. We apply Remark 1.15 in what follows. Now $(DD' - D'D)(x_1x_2, y) = \min\{DD'(x_1x_2, r_1),\ D'D(x_1x_2, r_2)\}$ (where $y = r_1 - r_2, DD'(x_1x_2, r_1) > 0$, and $D'D(x_1x_2, r_2) > 0$) $= \min\{\min\{D'(x_1x_2, w), D(w, r_1)\},\ \min\{D(x_1x_2,\ z), D'(z, r_2)\}\}$ (where $D'(x_1x_2, w) > 0, D(w, r_1) > 0$, $D(x_1x_2, z) > 0$, and $D'(z, r_2) > 0$) $\geq \min\{\min\{\min\{D'(x_2, w_2), D'(x_1, w_1)\}, D(w, r_1)\}, \min\{\min\{D(x_2, z_2), D(x_1, z_1)\}, D'(z, r_2)\}\}\}$ (where $w = x_1w_2 + x_2w_1, z = x_1z_2 + x_2z_1, D'(x_2, w_2) > 0$, $D'(x_1, w_1)\} > 0$, $D(w, r_1) > 0$, $D(x_2, z_2) > 0$, $D(x_1, z_1) > 0$, $D'(z, r_2) > 0$) $= \min\{\min\{\min\{D'(x_2, w_2),\ D'(x_1, w_1)\}, D(x_1w_2 + x_2w_1, r_1)\},\ \min\{\min\{D(x_2, z_2),\ D(x_1, z_1)\},\ D'(x_1z_2 + x_2z_1, r_2)\}\}\} \geq \min\{\min\{\min\{D'(x_2, w_2),\ D'(x_1, w_1)\},\ \min\{D(x_1w_2, a), D(x_2w_1, b)\}\}, \min\{\min\{D(x_2, z_2), D(x_1, z_1)\}, \min\{D'(x_1z_2, c), D'(x_2z_1,\ d)\}\}\}$(where $r_1 = a + b, r_2 = c + d$) $\geq \min\{\min\{\min\{D'(x_2, w_2), D'(x_1, w_1)\},\ \min\{\min\{D(x_1, z_1), D(w_2, u_2)\}, \min\{D(x_2, z_2), D(w_1, u_1)\}\}, \min\{\min\{D(x_2, z_2), D(x_1, z_1)\}, \min\{\min\{D'(x_1, w_1), D'(z_2, v_2)\}, \min\{D'(x_2, w_2), D'(\ z_1, v_1)\}\}\}$, (where $a = w_2z_1 + x_1u_2$, $b = w_1z_2 + x_2u_1$, $c = z_2w_1 + x_1v_2$, $d = z_1w_2 + v_1x_2$) $= \min\{\min\{\min\{D'(x_1, w_1), D(w_1, u_1)\}, \min\{D(x_1, z_1), D'(z_1, v_1)\}\},\ \min\{\min\{D'(x_2, w_2),\ D(w_2, u_2)\}, \min\{D(x_2, z_2), D'(z_2, v_2)\}\}\} = \min\{\min\{DD'(x_1, u_1),\ D'D(x_1, v_1)\}, \min\{DD'(x_2, u_2),\ D'D(x_2, v_2)\}\} = \min\{DD' - D'D)(x_1, s_1),\ (DD' - D'D)(x_2, s_2)\} = \sup\{\min\{(DD' - D'D)(x_1, y_1),\ (DD' - D'D)(x_2, y_2)\} | y = x_2y_1 + x_1y_2 \in A\}$, where $(DD' - D'D')(x_1, s_1) > 0, (DD' - D'D')(x_2, s_2) > 0$, $s_1 = u_1 - v_1$, and $s_2 = u_2 - v_2$.

We let $K[x]$ denote a polynomial ring over the field K. Let $T = K[x]/ < x^2 >$. Then $T = K[t]$, where t is the coset $x+ < x^2 >$. Thus $t^2 = 0$. Form the tensor product $B \otimes_K T$. Then $\forall z \in B \otimes_K T$, $\exists y_0, y_1 \in B$ such that $z = y_0 \otimes 1 + y_1 \otimes t$. Now $(y_0 \otimes 1 + y_1 \otimes t)(z_0 \otimes 1 + z_1 \otimes t) = y_0 z_0 \otimes 1 + (y_0 z_1 + y_1 z_0) \otimes t$, where $y_0, y_1, z_0, z_1 \in B$. $B \otimes_K T$ is called the *algebra of dual numbers* over B. Let D be a fuzzy derivation of A into B. Define $s : A \times (B \otimes_K T) \to [0,1]$ by $\forall x, x' \in A$ and $\forall y \in B$,

$$s(x, x' + yt) = \begin{cases} D(x,y) & \text{if } x = x' \text{ and } D(x,y) > 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (*)$$

We now show that s is a fuzzy (algebra) isomorphism of A into $B \otimes_K T$. In the following, $x_1, x_2, x \in A$ and $y_1, y_2, y \in B$. We write $x + yt$ for $x \otimes 1 + y \otimes t$.

$s(x, x + yt) > 0, s(x', x' + y't) > 0$, and $x = x' \Leftrightarrow D(x,y) > 0$ and $D(x', y') > 0$ and $x = x' \Rightarrow x = x'$ and $y = y' \Leftrightarrow x + yt = x' + y't$. Thus s is single-valued. Suppose that $s(x, x + yt) > 0$, $s(x', x' + y't) > 0$, and $x + yt = x' + y't$. Then $x = x'$ and $y = y'$. Hence s is one-to-one.

Now $s(x_1 + x_2, x_1 + x_2 + yt) = D(x_1 + x_2, y) \geq \sup\{\min\{D(x_1, y_1), D(x_2, y_2)\} | y = y_1 + y_2,\ y_1, y_2 \in B\} = \sup\{\min\{s(x_1, x_1 + y_1 t), s(x_2, x_2 + y_2 t)\} | y = y_1 + y_2,\ y_1, y_2 \in B\} = \sup\{\min\{s(x_1, x_1 + y_1 t),\ s(x_2, x_2 + y_2 t)\} | x_1 + x_2 + yt = x_1 + y_1 t + x_2 + y_2 t,\ y_1, y_2 \in B\}$. Also $s(x_1 x_2, x_1 x_2 + yt) = D(x_1 x_2, y) \geq \sup\{\min\{D(x_1, y_1), D(x_2, y_2)\} | y = x_2 y_1 + x_1 y_2, y_1, y_2 \in B\} = \sup\{\min\{s(x_1, x_1 + y_1 t), s(x_2, x_2 + y_2 t)\} | y = x_2 y_1 + x_1 y_2,\ y_1, y_2 \in B\} = \sup\{\min\{s(x_1, x_1 + y_1 t), s(x_2, x_2 + y_2 t)\} | x_1 x_2 + yt = (x_1 + y_1 t)(x_2 + y_2 t), y_1, y_2 \in B\}$ since $x_1 x_2 + yt = (x_1 + y_1 t)(x_2 + y_2 t) \Leftrightarrow y = x_2 y_1 + x_1 y_2$.

Let $k \in K$. Suppose $k \neq 0$. Then $s(kx, k(x + yt)) = D(kx, ky) = D(x,y) = s(x, x + yt)$. Suppose $k = 0$. Then $s(0x, 0(x + yt)) = D(0,0) \geq D(x,y) = s(x, x + yt)$.

Define $\pi : (B \otimes_K T) \times B \to [0,1]$ by $\forall x, x', y \in B$,

$$\pi(x+yt, x') = \begin{cases} 1 & \text{if } x \notin A \text{ and } x = x' \\ s(x', x+yt) & \text{if } x \in A \text{ and } x = x' \\ 0 & \text{if } x \neq x'. \end{cases}$$

Let $x \in A$. Then $(\pi \circ s)(x, x') = \sup\{\min\{s(x,z), \pi(z,x')\} | z \in B \otimes_K T\} = \min\{s(x, x+yt), \pi(x+yt, x')\} = s(x, x+yt) = \pi(x+yt, x) = D(x,y)$, where $D(x,y) > 0$ and $x = x'$. If $x \neq x'$, then $(\pi \circ s)(x,x') = 0$. That is, $\forall x \in A$, $(\pi \circ s)(x,x') > 0$ if and only if $x = x'$ and $(\pi \circ s)(x,x) = s(x, x+yt) = \pi(x+yt, x) = D(x,y)$ when $D(x,y) > 0$. Thus $\pi \circ s$ is an identity on A.

Conversely, let s be any fuzzy homomorphism of A into $B \otimes_K T$ such that $\pi \circ s$ is an identity on A. Define $D : A \times B \to [0,1]$ by $\forall (x,y) \in A \times B$, $D(x,y) = s(x, x+yt)$. Then D is single-valued since s is. Now $D(x_1+x_2, y) = s(x_1+x_2, x_1+x_2+yt) \geq \sup\{\min\{s(x_1, x_1+y_1t), \quad s(x_2, x_2+y_2t)\} | \ y = y_1+y_2, y_1, y_2 \in B\}$ $= \sup\{\min\{s(x_1, x_1+y_1t), s(x_2, x_2+y_2t) | x_1+x_2+y_t = x_1+y_1t+x_2+y_2t, y_1, y_2 \in B\} = \sup\{\min D(x_1,y_1), D(x_2,y_2)\} | y = y_1+y_2, y_1, y_2 \in B\}$. Let $k \in K$. Suppose $k \neq 0$. Then $D(kx, ky) = s(kx, k(x+yt)) = s(x, x+yt) = D(x,y))$. Suppose $k = 0$. Then $D(0x, 0y) = s(0x, 0(x+yt)) \geq s(x, x+yt) = D(x,y)$.

Also $D(x_1, x_2, y) = s(x_1x_2, x_1x_2+yt) \geq \sup\{\min\{s(x_1, x_1+y_1t), s(x_2, x_2+y_2t)\} | x_1x_2+yt = (x_1+y_1t)(x_2+y_2t), \ y_1, y_2 \in B\} = \sup\{\min\{ s(x_1, x_1+y_1t), \ s(x_2, x_2+y_2t)\} | y = x_2y_1+x_1y_2, \ y_1, y_2 \in B\} = \sup\{\min\{ D(x_1,y_1), D(x_2,y_2)\} | y = x_2y_1+x_1y_2, \quad y_1, y_2 \in B\}$ since $x_1x_2+yt = (x_1+y_1t)(x_2+y_2t) \Leftrightarrow y = x_2y_1+x_1y_2$. Hence D is a fuzzy derivation of A into B.

We have proved the following result.

Theorem 2.3. *Let A be a subalgebra of B and let D be a fuzzy derivation of A into B. If s is defined as in equation $(*)$, then s is a fuzzy isomorphism of A into the algebra of dual numbers $B \otimes_K T$ over B such that $\pi \circ s$ is an identity on A. Conversely, for any fuzzy homomorphism s*

of A into $B \otimes_K T$ satisfying this condition, there exist a fuzzy derivation D of A into B such that s is defined as in equation $(*)$ In terms of D.

3. PROPERTIES OF FUZZY DERIVATIONS

Definition 3.1. Let D he a fuzzy derivation of A into B. Then $c \in A$ is called a *D-constant* if $D(c, 0) > 0$.

Proposition 3.2. *Let D be a fuzzy derivation of A into B. Then the set C of all D-constants is a subalgebra of A.*

Proof. We have that $c \in A$ is a D-constant if and only if $s(c,c) > 0$. Now $0 \in C$ and so $C \neq \phi$. Let $c, d \in C$. Then $s(c-d, c-d) \geq \sup\{\min\{s(c,y_1), s(-d,y_2)\} | c-d = y_1 + y_2, \quad y_1, y_2 \in B\} \geq \min\{s(c,c), s(-d,-d)\} > 0$ by Proposition 1.4 (iii). Thus $c - d \in C$. Also $s(cd, cd) \geq \sup\{\min\{s(c,y_1), s(d,y_2)\} | cd = y_1 y_2, \quad y_1, y_2 \in B\} \geq \min\{s(c,c), s(d,d)\} > 0$. Hence $cd \in C$. Let $k \in K$. Then $s(kc, kc) = s(c,c) > 0$ if $k \neq 0$ and $s(kc, kc) = s(0,0) \geq s(c,c) > 0$ if $k = 0$. Thus $kc \in C$.

Proposition 3.3. *Let D be a fuzzy derivation of A into B. Let $C_t = \{c \in C | D(c, 0) \geq t\}$ and $A_t = \{a \in A | \exists b \in B, D(a,b) \geq t\}$ $\forall t \in [0, D(0,0)]$. Then C_t and A_t are subalgebras of $A \forall t \in [0, D(0,0)]$.*

Proof. $0 \in C_t$ and so $C_t \neq 0$. Let $c, d \in C_t$. Then $s(c+d, c+d) \geq \sup\{\min\{s(c,y_1), s(d,y_2)\} | c+d = y_1 + y_2, y_1, y_2 \in B\} \geq \min\{s(c,c), s(d,d)\} \geq t$. Thus $c+d \in C_t$. Also $s(cd, cd) \geq \sup\{\min\{s(c,y_1), s(d,y_2)\} | cd = y_1 y_2, y_1, y_2 \in B\} \geq \min\{s(c,c), s(d,d)\} \geq t$. Hence $cd \in C_t$. Let $k \in K$. Then $s(kc, kc) = s(c,c) \geq t$ if $k \neq 0$ and $s(kc, kc) = s(0,0) \geq s(c,c) \geq t$ if $k = 0$. Thus $kc \in C_t$. Hence $-c = -1c \in C_t$. A similar argument shows that A_t is a subalgebra of A.

Proposition 3.4. *Let D be a fuzzy derivation of A into B. Then $\forall a \in A$, $\forall b \in B$, and $\forall n \in N$, $D(a,b) > 0$ implies $D(a^n, na^{n-1}b) > 0$.*

Proof. We prove the result by induction on n. The case $n = 1$ is the hypothesis. Suppose the result is true for $n = k \geq 1$. Let $n = k + 1$.

We have $D(a^{k+1}, (k+1)a^k b) \geq \sup\{\min\{D(a, y_1), D(a^k, y_2)\} | (k+1)a^k b = ay_2 + a^k y_1\} \geq \min\{D(a, b), D(a^k, ka^{k-1}b)\} > 0$. Hence the desired result is true by induction.

Proposition 3.5. *Let D be a fuzzy derivation of A into B. Then $D(1,0) > 0$.*

Proof. $\exists\ b \in B$ such that $D(1, b) > 0$ and so $s(1, 1 + bt) > 0$. Thus $s(1, (1 + bt)(1 + bt)) \geq \min\{s(1, 1 + bt), s(1, 1 + bt)\} > 0$. Thus $1 + bt = (1 + bt)(1 + bt)$. Hence $b = b + b$. Thus $b = 0$. Therefore $D(1, 0) = s(1, 1) > 0$.

Proposition 3.6. *Let D be a fuzzy derivation of A into B. Let A be a field. Then $\forall a \in A$, $\forall b \in B$, and $\forall n \in N$, $D(a, b) > 0$ implies $D(a^{-n}, -na^{-n-1}b) > 0$, where $a \in A$, $a \neq 0$.*

Proof. Let $n = 1$. Then $s(a, a + bt) = D(a, b) > 0$. There exists $z \in B$ such that $s(a^{-1}, a^{-1} + zt) > 0$. Hence $s(1, (a + bt)(a^{-1} + zt)) \geq \min\{s(a, a + bt), s(a^{-1}, a^{-1} + zt)\} > 0$. Thus $1 = (a + bt)(a^{-1} + zt)$. Hence $az + ba^{-1} = 0$. Thus $z = -a^{-2}b$. Hence $D(a^{-1}, -a^{-2}b) > 0$. Suppose the result is true for $n = k \geq 1$. Let $n = k + 1$. We have $D(a^{-k-1}, (-k-1)a^{-k-2}b) \geq \sup\{\min\{D(a^{-1}, y_1), D(a^{-k}, y_2)\} | (-k-1)a^{-k-2}b = a^{-1}y_2 + a^{-k}y_1\} \geq \min\{D(a^{-1}, -a^{-2}b), D(a^{-k}, -ka^{-k-1}b)\} > 0$. Hence the desired result is true by induction.

Theorem 3.7. *Let D be a fuzzy function from A into B. $\forall t \in [0, D(0, 0)]$, define the function D_t of A_t into B as follows: $\forall (a, b) \in A_t \times B, D_t(a) = b$ if $D(a, b) > 0$. Then the following conditions are equivalent.*

(1) D is a fuzzy derivation of A into B;

(2) $D(0, 0) \geq D(a, b) \forall (a, b) \in A \times B$ and D_t is a derivation of A_t into B $\forall t \in [0, D(0, 0)]$.

Proof. (1) $\Rightarrow$ (2): It follows easily that D_t is a linear transformation of A into B. By Definition 1.3 (ii), $D(0, 0) \geq D(a, b)\quad \forall (a, b) \in A \times B$. Let $a_1, a_2 \in A_t$. There exists $b_1, b_2 \in B$ such that $D(a_1, b_1) > 0$ and

$D(a_2, b_2) > 0$. Let $b = a_1b_2 + a_2b_1$. Then $D(a_1a_2, b) \geq \min\{D(a_1, b_1), D(a_2, b_2)\} > 0$. Since $a_1a_2 \in A_t$ by Proposition 3.3, $D_t(a_1a_2) = a_1b_2 + a_2b_1$. Since $a_1, a_2 \in A_t$, $D_t(a_i) = b_i, i = 1, 2$. Thus $D_t(a_1a_2) = a_1D_t(a_2) + a_2D_t(a_1)$.

(2) $\Rightarrow$ (1): Let $a_1, a_2 \in A$. Then $\exists b_1, b_2 \in B$ such that $D(a_i, b_i) > 0$ for $i = 1, 2$. Let $D(a_i, b_i) = s_i$ for $i = 1, 2$. Let $t = \min\{s_1, s_2\}$. Then D_t is a derivation since $s_1, s_2 \leq D(0, 0)$. Now $a_1, a_2 \in A_t$ and so $a_1a_2 \in A_t$. Also $D_t(a_1a_2) = a_1D_t(a_2) + a_2D_t(a_1)$. By the definition of D_t, $D(a_1a_2, a_1D_t(a_2) + a_2D_t(a_1)) > 0$. Also by the definition of $D_t, D_t(a_i) = b_i, i = 1, 2$. Now $a_1, a_2 \in A_r$ implies $a_1a_2 \in A_r$, $\forall r \in [0, D(0, 0)]$. Thus $D(a_1a_2, a_1D_t(a_2) + a_2D_t(a_1)) \geq \min\{D(a_1, b_1), D(a_2, b_2)\}$. One can also show that D is a fuzzy linear operator. Thus D is a fuzzy derivation of A into B.

From Theorem 3.7, we see that a fuzzy derivation D of A into B induces, in a natural way, a crisp derivation of A into B, namely D_0.

Corollary 3.8. *Define the function f of $\mathcal{FL}(A, B)$ into $\mathcal{L}(A, B)$ by for all $T \in \mathcal{FL}(A, B)$, $f(T)(a) = b$ if $T(a, b) > 0$. Then f preserves addition and scalar multiplication. In fact, f maps $\mathcal{FD}(A, B)$ onto $\mathcal{D}(A, B)$, where $\mathcal{D}(A, B)$ is the subspace of $\mathcal{L}(A, B)$ of all derivations of A into B.*

Proof. That f actually maps $\mathcal{FL}(A, B)$ into $\mathcal{L}(A, B)$ follows easily. By Theorem 3.7, f maps $\mathcal{FD}(A, B)$ into $\mathcal{D}(A, B)$. Let $D_0 \in \mathcal{D}(A, B)$. Define D by $\forall (a, b) \in A \times B$, $D(a, b) = 1$ if $D_0(a) = b$ and $D(a, b) = 0$ otherwise. Then clearly $D \in \mathcal{FD}(\mathcal{A}, \mathcal{B})$ and $f(D) = D_0$. Hence f maps $\mathcal{FD}(A, B)$ onto $\mathcal{D}(A, B)$. In a similar manner, we see that f maps $\mathcal{FL}(A, B)$ onto $\mathcal{L}(A, B)$. Hence it remains to show that f preserves addition and scalar multiplication. Let $T, T' \in \mathcal{FL}(A, B)$. Then $f(T + T')(a) = b \Leftrightarrow (T + T')(a, b) > 0 \Leftrightarrow \min\{T(a, b_1), T'(a, b_2)\} > 0$, where $b = b_1 + b_2 \Leftrightarrow T(a, b_1) > 0, T'(a, b_2) > 0$, where $b = b_1 + b_2 \Leftrightarrow f(T)(a) = b_1$ and $f(T')(a) = b_2$, where $b = b_1 + b_2 \Leftrightarrow (f(T) + f(T'))(a) = b$, where $b = b_1 + b_2$. Therefore $f(T + T') = f(T) + f(T')$. Let $k \in K$, $k \neq 0$. Then $f(kT)(a) = b \Leftrightarrow (kT)(a, b) > 0 \Leftrightarrow T(a, k^{-1}b) > 0 \Leftrightarrow f(T)(a) = k^{-1}b \Leftrightarrow k(f(T))(a) = b$. Thus $f(kT) = kf(T)$. Suppose that $k = 0$. Then $f(0T)(a) = b \Leftrightarrow f(\Theta)(a) = b \Leftrightarrow \Theta(a, b) > 0 \Leftrightarrow b = 0 \Leftrightarrow 0f(T)(a) = b$, where $\Theta(a,b) = 1$ if $b = 0$ and $\Theta(a,b) = 0$ otherwise. Thus $f(0T) = 0f(T)$.

Suppose that A and B are fields. Define $\lambda(a) = D(a,b)$, where $D(a,b) > 0$. Then the level set $\lambda_t = A_t$. Define $\gamma(a) = D(a,0)$. Then $\gamma(a) > 0$ if and only if $a \in C$. Thus $\gamma^* = C$. Also the level set $\gamma_t = C_t$. $D(-a,-b) = D(-1(a,b)) \geq D(a,b) = D(-1(-a,-b)) \geq D(-a,-b)$. Thus $D(-a,-b) = D(a,b)$. By Proposition 3.6, $D(c,0) > 0$ implies $D(c^{-1},0) > 0$. Hence C is a field.

By Proposition 3.5, $D(k,0) = D(k(1,0)) \geq D(1,0) > 0$, where $k \in K$. Hence $K \subseteq C$. In fact, $D(k,0) = D(1,0)\ \forall k \in K, k \neq 0$, since $D(1,0) = D(k^{-1}k,0) = D(k^{-1}(k,0)) \geq D(k,0)$. Thus γ is a constant on $K\backslash\{0\}$.

Suppose that A has characteristic $p > 0$. By Proposition 3.4, $D(a^p,0) > 0\ \forall a \in A$ since $D(a,b) > 0$ for some $b \in B$. Hence $A^p \subseteq C$.

We now show that if A and B are fields, it is not necessarily the case that the A_t of Proposition 3.3 are fields.

Example 3.9. Let $A = B = K(x)$, where K is a field and x is an indeterminate over K. Let P denote the set of all elements of $K[x]$ with 0 constant term. If $u \in A$, we let u' denote the formal derivative of u with respect to x. Define the fuzzy subset D of $A \times B$ as follows:

$D(0,0) = 1, D(u,u') = 3/4$ if $u \in P\backslash\{0\}$, $D(u,v) = 1/2$ if $u \in A\backslash P$ and $v = u'$, and $D(u,v) = 0$ otherwise. Then $D(k,0) = 1/2\ \forall k \in K\backslash\{0\}$. We see that P is an algebra over K, but $K \nsubseteq P$. Let $u,v \in A$. If $u+v \notin P$, then either $u \notin P$ or $v \notin P$ and so $D(u+v,(u+v)') = 1/2 = \min\{D(u,u'),D(v,v')\}$ since $(u+v)' = u'+v'$. If $uv \notin P$, then either $u \notin P$ or $v \notin P$ and so $D(uv,(uv)') = 1/2 = \{D(u,u'),D(v,v')\}$ since $(uv)' = uv'+vu'$. If $u+v \in P$, then $D(u+v,(u+v)') \geq \min\{D(u,u'),D(v,v')\}$. If $uv \in P$, then $D(uv,(uv)') \geq \min\{D(u,u'),D(v,v')\}$. We also have that $D(u,u') = D(-u,(-u)')$ since $u \in P$ if and only if $-u \in P$ and $(-u)' = -u'$. Let $k \in K\backslash\{0\}$. Then $ku \in P$ if and only if $u \in P$. Hence $D(ku,ku') = D(u,u')$. Thus D is a fuzzy derivation of A into B. Now $A_{3/4} = P$ and so $K \nsubseteq A_{3/4}$. $A_{3/4}$ is not a subfield of A.

Theorem 3.10. *Suppose that A and B are fields. Let D be a fuzzy derivation of A into B. Let $\alpha = \sup\{D(a,b) | (a,b) \in A \times B \backslash \{(0,0)\}$. Then conditions (i), (ii), and (iii) are equivalent. If A/K is algebraic, then all four conditions are equivalent.*

(i) $K \subseteq C_t \ \forall t \in [0, \alpha]$;

(ii) $K \subseteq A_t \ \forall t \in [0, \alpha]$;

(iii) $D(1,0) \geq D(a,b) \ \forall (a,b) \in A \times B \backslash \{(0,0)\}$;

(iv) C_t and A_t are intermediate fields of $A/K \forall t \in [0, \alpha]$.

Proof. (i) $\Rightarrow$ (ii): Immediate since $C_t \subseteq A_t \forall t \in [0, \alpha]$.

(ii) $\Rightarrow$ (iii): Let $(a,b) \in A \times B \backslash \{(0,0)\}$ and let $D((a,b)) = t$. Since $K \subseteq A_t$, $1 \in A_t$. Thus $D(1,0) \geq t$.

(iii) $\Rightarrow$ (i): Since $D(k,0) = D(1,0) \ \forall k \in K \backslash \{0\}$, $D(k,0) \geq D(a,b)$ $\forall (a,b) \in A \times B\{(0,0)\}$ and $\forall k \in K$ and so $D(k,0) \geq D(c,0) \ \forall k \in K$ and $\forall c \in C$. Thus $K \subseteq C_t \ \ \forall t \in [0, \alpha]$, i.e., (i) holds.

(iv) $\Rightarrow$ (i): Immediate.

(i) $\Rightarrow$ (iv): Suppose that A/K is algebraic. Then since and C_t and A_t are rings containing K, we have that C_t and A_t are fields.

Since $\gamma_t = C_t$ and $\lambda_t = A_t \forall t \in [0, D(0,0)]$, we have that γ and λ are fuzzy subalgebras of A. If $D(1,0) \geq D(a,b) \ \forall (a,b) \in A \times B \backslash \{(0,0)\}$ and A/K is an algebraic field extension, then Theorem 3.10 shows that γ_t and λ_t are intermediate fields of $A/K \ \forall t \in [0, \alpha]$. Thus Theorem 3.10 gives the property needed to bring fuzzy derivation theory to the study of fields, namely that $D(1,0) \geq D(a,b) \ \forall (a,b) \in A \times B \backslash \{(0,0)\}$.

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