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GEOMETRY OF DEGENERATE HYPERSURFACES

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ABSTRACT. In this paper we investigate the geometry of degenerate hypersurfaces of semi-Riemannian manifolds. It is found that such hypersurfaces carry a special type of distribution (the screen distribution) which plays an important role in determining the geometry of the hypersurface. A fundamental existence theorem for degenerate hypersurfaces is proved and several results on the degenerate hypersurfaces of Lorentz space are obtained. In particular, it is shown that the geometry of degenerate hypersurfaces of Lorentz space mainly reduces to the study of the geometry of the Riemannian foliation defined by the canonical screen distribution.

INTRODUCTION

The geometry of submanifolds in manifolds endowed with some geometrical structures has been intensively studied and several important results have been obtained (see [2],[7], [11], [13], [30], [31] and the references therein). In case the ambient space is a semi-Riemannian manifold, degenerate submanifolds have been introduced and investigated (cf. [3]-[7], [8],[10], [14]-[16],[23]-[25]). The study of a degenerate submanifold is essentially different from the one of a non-degenerate submanifold because of the lack of a canonical transversal bundle which has to replace the normal bundle from the classical theory of Riemannian submanifolds.

The purpose of the present paper is to present to the reader the author's point of view with respect to the differential geometry of degenerate hypersurfaces. In the first section we introduce the reader to the general theory of semi-Riemannian and degenerate manifolds. In section 2 we construct the transversal vector bundle (see [4] for the general case), give examples and define the induced geometric objects on a degenerate

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hypersurface. The main ingredient in our study is the screen distribution on a degenerate hypersurface. In case the ambient space is either a time-orientable Lorentz manifold or a semi- Euclidean space, we construct a canonical screen distribution. In section 3 we present the Gauss - Codazzi structure equations and prove the Fundamental Theorem for degenerate immersions. Section 4 is devoted to the study of totally geodesic and totally umbilical degenerate hypersurfaces. The canonical screen distribution is used in the last section for new results on degenerate hypersurfaces of a Lorentz space. Here we show that the study of differential geometry of a degenerate hypersurface M in a Lorentz space R_1^{m+2}, mainly reduces to the study of differential geometry of the Riemannian foliation defined by the canonical screen distribution $S(TM)$.

1. SEMI-RIEMANNIAN MANIFOLDS AND DEGENERATE MANIFOLDS

Let $\bar{M}$ be a real $(m+2)$-dimensional smooth manifold and $\bar{g}$ a symmetric tensor field of type (0,2) on $\bar{M}$. Thus $\bar{g}$ assigns smoothly to each point x of $\bar{M}$, a symmetric bilinear form $\bar{g}_x$ on the tangent space $T_x\bar{M}$. The dimension of the largest subspace $W_x \subset T_x\bar{M}$, on which $\bar{g}_x$ is negative definite is called the *index* of $\bar{g}_x$ on $T_x\bar{M}$. In the present paper we suppose $\bar{g}_x$ is non-degenerate on $T_x\bar{M}$ and the index of $\bar{g}_x$ is the same for all $x \in \bar{M}$. Thus each $T_x\bar{M}$ becomes a $(m+2)$- dimensional semi-Euclidean space. The tensor field $\bar{g}$ satisfying the above conditions is called a *semi-Riemannian (pseudo-Riemannian) metric* and $\bar{M}$ endowed with $\bar{g}$ is called a *semi-Riemannian (pseudo-Riemannian) manifold.*

The geometry of semi-Riemannian manifolds and its applications to relativity theory is very well presented in [19]. We only introduce here the main concepts from the geometry of semi-Riemannian manifolds, which are necessary for developing a theory of degenerate hypersurfaces.

As $T_x\bar{M}$ is a semi-Euclidean space, a tangent vector $u \in T_x\bar{M}$ is said to be

spacelike, if $\bar{g}_x(u,u) > 0 \quad \text{or} \quad u = 0,$

timelike, if $\bar{g}_x(u,u) < 0$,

lightlike (null), if $\bar{g}_x(u,u) = 0$ and $u \neq 0$.

We keep the same terminology for vector fields on $\bar{M}$ and even for vector fields on degenerate hypersurfaces.

Suppose q is the index of $\bar{M}$, that is, q is the common value of the index of $\bar{g}_x$ for any $x \in \bar{M}$. In case $q = 1$, $\bar{M}$ (resp. $\bar{g}$) is called a *Lorentz manifold* (resp. *Lorentz metric*). In the present paper we suppose $0 < q < m+2$, that is, $\bar{g}$ cannot be positive or negative definite.

Denote by $\mathcal{F}(\bar{M})$ the algebra of smooth real functions on $\bar{M}$ and by $\Gamma(T\bar{M})$ the $\mathcal{F}(\bar{M})$-module of smooth vector fields on $\bar{M}$. Similarly, for any vector bundle E over $\bar{M}$, denote by $\Gamma(E)$ the $\mathcal{F}(\bar{M})$-module of smooth sections of E. Throughout the paper we keep the above notation for any other vector bundle.

Next, consider a linear connection $\bar{\nabla}$ on the semi-Riemannian manifold $(\bar{M}, \bar{g})$. We say that $\bar{\nabla}$ is a *metric connection (Riemannian connection)* if the metric tensor field $\bar{g}$ is parallel with respect to $\bar{\nabla}$, i.e., if we have

$$(\bar{\nabla}_X \bar{g})(Y,Z) = X(\bar{g}(Y,Z)) - \bar{g}(\bar{\nabla}_X Y, Z) - \bar{g}(Y, \bar{\nabla}_X Z) = 0, \tag{1.1}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. The following result is of great importance for the geometry of semi-Riemannian manifolds.

Theorem 1.1 ([19], p. 61). *On a semi-Riemannian manifold there exists a unique torsion-free metric connection.*

The metric connection from the above theorem is called the *Levi-Civita connection* and it is given by

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_X Y, Z) &= X(\bar{g}(Y,Z)) + Y(\bar{g}(X,Z)) - Z(\bar{g}(X,Y)) + \bar{g}([X,Y],Z) \\ &\quad + \bar{g}([Z,X],Y) - \bar{g}([Y,Z],X), \quad \forall\, X,Y,Z \in \Gamma(T\bar{M}). \end{aligned} \tag{1.2}$$

In case $\bar{M}$ is a semi-Riemannian manifold of constant sectional curvature c it is denoted by $\bar{M}(c)$. Then the curvature tensor field $\bar{R}$ of $\bar{M}(c)$

is given by

$$(1.3)\quad \bar{R}(X,Y)Z = c(\bar{g}(Y,Z)X - \bar{g}(X,Z)Y), \quad \forall\, X,Y,Z \in \Gamma(T\bar{M}).$$

Example 1.1. On R^{m+2} we consider the semi-Euclidean metric

$$(1.4)\qquad \bar{g}(x,y) = -\sum_{i=0}^{q-1} x^i y^i + \sum_{a=q}^{m+1} x^a y^a,$$

of index $0 < q < m+2$. Denote by R_q^{m+2} the semi-Euclidean space $(R^{m+2}, \bar{g})$ with $\bar{g}$ given by (1.4). The Levi-Civita connectin on R_q^{m+2} is defined by

$$(1.5)\qquad \bar{\nabla}_X Y = \sum_{A=0}^{m+1} X(Y^A)\frac{\partial}{\partial x^A},$$

where $Y = Y^A(\partial/\partial x^A)$. Finally, the curvature tensor field $\bar{R}$ of $\bar{\nabla}$ vanishes and therefore R_q^{m+2} is of constant curvature $c = 0$.

Example 1.2. In a semi-Euclidean space R_q^{m+2} define the *pseudosphere* of radius $r > 0$ by

$$S_q^{m+1}(r) = \left\{ x \in R_q^{m+2} : -\sum_{i=0}^{q-1}(x^i)^2 + \sum_{a=q}^{m+1}(a^a)^2 = r^2 \right\},$$

and the *pseudohyperbolic space* of radius $r > 0$ by

$$H_{q-1}^{m+1}(r) = \left\{ x \in R_q^{m+2} : -\sum_{i=0}^{q-1}(x^i)^2 + \sum_{a=q}^{m+1}(a^a)^2 = -r^2 \right\}.$$

Both $S_q^{m+1}(r)$ and $H_{q-1}^{m+1}(r)$ are totally umbilical semi- Riemannian submanifolds of index q and $q-1$ of constant curvature $c = 1/r^2$ and $c = -1/r^2$, respectively.

Next, suppose M is a real $(m+1)$-dimensional smooth manifold and g is a symmetric tensor field of type (0,2) on M such that g_x is of constant index q on T_xM for any $x \in M$. Define the *radical subspace* of T_xM by (cf. [1])

$$Rad\ T_xM = \{v \in T_xM : g(v,u) = 0, \quad \forall\, u \in T_xM\}$$

.

We say that (M, g) is an *r-degenerate manifold* if the mapping $Rad\,TM$ assigns to each $x \in M$ the radical subspace $Rad\,T_xM$ defines a smooth distribution of rank $r > 0$ on M. In this case $Rad\,TM$ is called the *radical distribution* on M . By the above definition we have

$$(1.6)\qquad g(V, X) = 0, \quad \forall\, X \in \Gamma(TM), \quad V \in \Gamma(Rad\,TM).$$

Moreover, it is easy to see that (M, g) is r-degenerate if and only if g has a constant rank $m - r + 1$ on M.

In literature such manifolds have been studied under several names: singular Riemannian spaces ([14],[18],[28],[29]), degenerate Riemannian manifolds ([12], [20]), degenerate pseudo-Riemannian manifolds ([26]), degenerate manifolds ([17]), isotropic spaces ([25]) and lightlike manifolds ([7]).

Now, suppose $Rad\,TM$ is an integrable distribution. Then there exists a coordinate system $(\mathcal{U}; x^1, \ldots, x^{m+1})$ on M such that (x^α), $\alpha \in \{1, \ldots, r\}$ are the coordinates on a leaf M' of $Rad\,TM$ and $x^i = c^i$, $i \in \{r+1, \ldots, m+1\}$ are the local equations of M'. As g is r-degenerate on TM, from (1.6) we derive

$$g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) = g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^i}\right) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\right) = 0.$$

Thus the matrix of g with respect to the natural frame field becomes

$$(1.7)\quad [g] = \begin{bmatrix} O_{r,r} & O_{r,m-r+1} \\ O_{m-r+1,r} & g_{ij}(x^1, \ldots, x^{m+1}) \end{bmatrix}, \quad i, j \in \{r+1, \ldots, m+1\}.$$

In case, with respect to the above coordinate system we have

$$(1.8)\qquad \frac{\partial g_{ij}}{\partial x^\alpha} = 0, \quad \alpha \in \{1, \ldots, r\}, \quad i, j \in \{r+1, \ldots, m+1\},$$

we say that M is a *Reinhart degenerate manifold.* We use the name of Reinhart because the class of Riemannian foliations satisfying (1.8) *(bundle-like metrics)* introduced in 1959 by Reinhart [21], were later named as Reinhart spaces (cf. [27]).

A vector field X on the degenerate manifold M is said to be a *Killing vector field* if

$$(1.9)\quad (L_X g)(Y,Z) = X(g(Y,Z)) - g([X,Y],Z) - g([X,Z],Y) = 0$$

for any $Y, Z \in \Gamma(TM)$. A distribution D on M is said to be a *Killing distribution* if each vector field that belongs to D is a Killing vector field.

An important research matter in the geometry of degenerate manifolds is the study of the existence of some particular linear connections on such manifolds. In this respect we state the following result.

Theorem 1.2. *Let (M,g) be a degenerate manifold. Then the following assertions are equivalent:*

(i) (M,g) is a Reinhart degenerate manifold.

(ii) $Rad\,TM$ is a Killing distribution.

(iii) There exists a torsion-free linear connection ∇ on M such that g is parallel with respect to ∇.

Proof. (i) $\Rightarrow$ (ii). Suppose M is a Reinhart degenerate manifold. As $Rad\,TM$ is integrable we may consider a coordinate system $(\mathcal{U}; x^1, \ldots, x^{m+1})$ such that any $X \in \Gamma(Rad\,TM)$ is locally expressed as $X = X^\alpha(\partial/\partial x^\alpha)$. In this way, $L_X g = 0$ is equivalent to

$$(1.10)\quad X^\alpha\left\{\frac{\partial g(Y,Z)}{\partial x^\alpha} - g([\partial/\partial x^\alpha, Y], Z) - g([\partial/\partial x^\alpha, Z], Y)\right\} = 0,$$

for any $Y, Z \in \Gamma(TM)$. By using (1.6) and taking into account that $Rad\ TM$ is integrable, it is easy to check that when at least one of vector fields Y and Z belongs to $Rad\,TM$, (1.10) is identically satisfied. Now take $Y = \partial/\partial x^i$ and $Z = \partial/\partial x^j$, $i, j \in \{r+1, \ldots, m+1\}$, and (1.10) follows by using (1.8). Hence $Rad\,TM$ is a Killing distribution.

(ii) $\Rightarrow$ (i). Suppose $Rad\ TM$ is a Killing distribution, that is, (1.9) holds for any $X \in \Gamma(Rad\ TM)$ and $Y, Z \in \Gamma(TM)$. Consider $Y \in \Gamma(Rad\,TM)$ in (1.9) and by using (1.6) obtain $g([X,Y],Z) = 0$, for any

$Z \in \Gamma(TM)$. Hence $[X, Y] \in \Gamma(Rad\, TM)$, that is, $Rad\, TM$ is involutive, and by Frobenius theorem, it is integrable. Finally, take $X = \partial/\partial x^\alpha \in \Gamma(Rad\, TM)$, $Y = \partial/\partial x^i$ and $Z = \partial/\partial x^j$ in (1.9) and obtain (1.8). Hence (M, g) is a Reinhart degenerate manifold.

(iii) $\Rightarrow$ (ii) Suppose there exists a torsion-free linear connection ∇ on M and g is parallel with respect to ∇. Then, by using (1.1), we obtain

$$\begin{aligned}(L_X g)(Y, Z) &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = Y(g(X, Z)) + Z(g(X, Y)) \\ &\quad -g(X, \nabla_Y Z) - g(X, \nabla_Z Y) = 0,\end{aligned}$$

for any $X \in \Gamma(Rad\, TM)$ and $Y, Z \in \Gamma(TM)$. Hence $Rad\, TM$ is a Killing distribution on M.

(ii) $\Rightarrow$ (iii). As (ii) is satisfied, from the proof of (ii) $\Rightarrow$ (i) it follows that $Rad\, TM$ is integrable. Consider $Rad\, TM$ as a $(m + 1 + r)$-dimensional manifold with local coordinates $(x^\alpha, x^i, y^\alpha)$, where (x^α, x^i) are the local coordinates on M induced by the foliation determined by $Rad\, TM$ and (y^α) are coordinates on fibres of vector bundle $Rad\, TM$. Thus the coordinate transformations on $Rad\, TM$ are given by

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^1, \ldots, x^{m+1}), \tilde{x}^i = \tilde{x}^i(x^{r+1}, \ldots, x^{m+1}), \tilde{y}^\alpha = B^\alpha_\beta(x) y^\beta,$$

where $B^\alpha_\beta(x) = \partial \tilde{x}^\alpha / \partial x^\beta$. Hence we obtain

$$\frac{\partial}{\partial y^\alpha} = B^\beta_\alpha(x) \frac{\partial}{\partial \tilde{y}^\beta}.$$

Thus there exists a vector bundle NM over M, locally spanned by $\{\partial/\partial y^\alpha\}$, $\alpha \in \{1, \ldots, r\}$, and transversal to TM, i.e., we have

$$T(Rad\, TM)_{|M} = TM \oplus NM. \tag{1.11}$$

Next, as M is assumed to be paracompact, we consider a Riemannian metric G on M and a complementary orthogonal distribution $S(TM)$ to $Rad\, TM$ in TM with respect to G. Thus (1.11) becomes

$$T(Rad\, TM)_{|M} = S(TM) \oplus Rad\, TM \oplus NM. \tag{1.12}$$

We should note that NM and $Rad\,TM$ are vector bundles of rank r over M such that the transition matrices from $\{\partial/\partial y^\alpha\}$ to $\{\partial/\partial \tilde{y}^\beta\}$ and from $\{\partial/\partial x^\alpha\}$ to $\{\partial/\partial \tilde{x}^\beta\}$ are the same. Hence any section $N = N^\alpha(\partial/\partial y^\alpha)$ of NM defines a section $N^* = N^\alpha(\partial/\partial x^\alpha)$ of $Rad\ TM$. Now, denote by A, B and C the projection morphisms of $T(Rad\,TM)_{|M}$ on $S(TM)$, $Rad\,TM$ and NM respectively, and define

$$(1.13)\quad \bar{g}(\bar{X},\bar{Y}) = g(A\bar{X},A\bar{Y}) + G(B\bar{X},(C\bar{Y})^*) + G(B\bar{Y},(C\bar{X})^*),$$

for any $\bar{X},\bar{Y} \in \Gamma(T(Rad\ TM)_{|M})$. It is easy to verify that $\bar{g}$ is a semi-Riemannian metric on the manifold $Rad\,TM$ and the degenerate metric g is the restriction of $\bar{g}$ to $\Gamma(TM)$. Denote by $\bar{\nabla}$ the Levi-Civita connection on $(Rad\,TM,\bar{g})$ and set

$$(1.14)\qquad \bar{\nabla}_X Y = \nabla_X Y + B^\alpha(X,Y)\frac{\partial}{\partial y^\alpha},\quad \forall\, X,Y \in \Gamma(TM),$$

where $\nabla_X Y \in \Gamma(TM)$ and $B^\alpha(X,Y) \in \mathcal{F}(M)$. It follows that ∇ is a torsion-free linear connection on M and B^α are symmetric bilinear forms on $\Gamma(TM)$. Moreover, by using (1.9), (1.13) and (1.14), and taking into account that $\bar{g}$ is parallel with respect to $\bar{\nabla}$ we obtain

$$0 = (L_X g)(Y,Z) = -\bar{g}(X,\bar{\nabla}_Y Z + \bar{\nabla}_Z Y) = -2B^\alpha(Y,Z)G\left(X,\frac{\partial}{\partial x^\alpha}\right),$$

for any $X \in \Gamma(Rad\ TM)$ and $Y,Z \in \Gamma(TM)$. Since $r > 0$ and G is a Riemannian metric on the distribution $Rad\,TM$ we deduce $B^\alpha(Y,Z) = 0$, for $\alpha \in \{1,\ldots,r\}$. It follows $\bar{\nabla}_Y Z = \nabla_Y Z$, that is g is parallel with respect to ∇. This completes the proof of the theorem.

We close this section with some remarks on the assertions of Theorem 1.2. The equivalence of (ii) and (iii) was first proved in [20] by using the theory of G- structures. The class of linear connections from (iii) was first considered in [18]. The quadratic forms satisfying (1.8) were studied in [9] and [22]. Finally, we consider the vector bundle $TM^* = TM/Rad\,TM$ and the canonical projection $p : TM \to TM^*$. Then on TM^* there exists a semi- Riemannian metric g^* defined by

$$g^*(pX,pY) = g(X,Y),\ \forall\ X,Y \in \Gamma(TM).$$

In [17] was proved the equivalence of (ii) with

(iv) *There exists a unique metric connection* ∇^* *on* TM^* *such that*

$$\nabla^*_X pY - \nabla^*_Y pX - p([X,Y]) = 0, \quad \forall\, X,Y \in \Gamma(TM).$$

It seems to the author that results from the theory of Riemannian foliations might be of interest for further research into degenerate manifolds. Actually, Vránceanu [29] has already applied his theory of non-holonomic spaces to the study of some invariants in the geometry of degenerate manifolds.

2. THE INDUCED GEOMETRIC OBJECTS ON A DEGENERATE HYPERSURFACE

Let $(\bar{M}, \bar{g})$ be a $(m+2)$-dimensional semi-Riemannian manifold of index $0 < q < m+2$ and M be a hyprsurface of $\bar{M}$. Then $\bar{g}$ induces a symmetric tensor field g of type (0,2) on M. Suppose (M,g) is a degenerate manifold, that is, *Rad* TM defines a distribution of rank $r > 0$ on M. As usual, for any $x \in M$ define

$$T_xM^\perp = \{v \in T_x\bar{M} : \bar{g}(v,u) = 0, \quad \forall\, u \in T_xM\},$$

and consider the vector bundle

$$TM^\perp = \bigcup_{x\in M} T_xM^\perp.$$

It is easy to see that M is degenerate if and only if *Rad* $TM = TM^\perp$. Therefore, in this case $r = 1$, and $TM^\perp$ (the former normal bundle for a non- degenerate hypersurface) becomes a distribution on M and rank $g = m$ on M.

Now suppose M is locally given by the equation

$$F(x^0, \dots, x^{m+1}) = 0, \tag{2.1}$$

where F is differentiable on a domain $D \subset R^{m+2}$ and rank $[F'_{x^0}, \dots, F'_{x^{m+1}}] = 1$. Denote $\bar{g}_{AB} = \bar{g}(\partial/\partial x^A, \partial/\partial x^B)$ and consider the gradient vector field of F defined by

$$grad\ F = \bar{g}^{AB} F'_{x^A} F'_{x^B}, \tag{2.2}$$

where $\bar{g}^{AB}$ are the entries of the inverse matrix of $[\bar{g}_{AB}]$. As $grad\ F$ is normal to M we conclude that M is degenerate if and only if $grad\ F$ at any $x \in M$ lies in T_xM, i.e., $(grad\ F)_x$ is a null vector with respect to $\bar{g}$. Thus by using (2.2) we obtain the following characterization for degenerate hypersurfaces.

Theorem 2.1. *M is a degenerate hypersurface of $\bar{M}$ if and only if on any coordinate neighborhood $\mathcal{U} \subset M$ on which M is given by (2.1), F satisfies the partial differential equation*

$$\bar{g}^{AB} F'_{x^A} F'_{x^B} = 0, \tag{2.3}$$

at any point of M.

In order to study the geometry of the degenerate hypersurface M we need a complementary vector bundle to TM in $T\bar{M}$ which is going to replace $TM^{\perp}$ from the classical theory of Riemannian hypersurfaces. To this end we consider a complementary vector bundle $S(TM)$ to $TM^{\perp}$ in TM, i.e., we have

$$TM = S(TM) \perp TM^{\perp}. \tag{2.4}$$

Clearly $S(TM)$ is also orthogonal to $TM^{\perp}$. Throughout the paper $\perp$ and $\oplus$ will denote orthogonal direct sum and a direct but not orthogonal sum respectively. As M is assumed paracompact, there always exists $S(TM)$. Moreover, we show later in this section how to construct a canonical distribution on some special degenerate hypersurfaces. Motivated by the fact that the lightlike rays on the null cone (see Example 2.1) lie on $TM^{\perp}$ and hence fibres of $S(TM)$ are visualized as transversal screens to these rays, we call $S(TM)$ a *screen distribution.*

It is easy to verify that $S(TM)$ is non-degenerate and therefore we have

$$T\bar{M}_{|M} = S(TM) \perp S(TM)^{\perp}, \tag{2.5}$$

where $S(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\bar{M}_{|M}$. Moreover, $S(TM)^{\perp}$ is a vector bundle of rank 2 and $TM^{\perp}$ is a vector subbundle of $S(TM)^{\perp}$. Consider a complementary vector bundle F to $TM^{\perp}$ in $S(TM)^{\perp}$ and take sections $V \in \Gamma(F_{|\mathcal{U}})$ and $\xi \in \Gamma(TM^{\perp}_{|\mathcal{U}})$, where $\mathcal{U}$ is a coordinate neighborhood of M. Note that $\bar{g}(\xi, V) \neq 0$, otherwise $S(TM)^{\perp}$ would be degenerate on $\mathcal{U}$. Finally, define the vector field

$$N = \frac{1}{\bar{g}(\xi, v)} \left\{ V - \frac{\bar{g}(V,V)}{2\bar{g}(\xi, V)} \xi \right\}, \tag{2.6}$$

on $\mathcal{U}$. By direct calculation, it follows that

$$g(N,N) = g(N,W) = 0, \quad \forall\, W \in \Gamma(S(TM)_{|\mathcal{U}}), \tag{2.7}$$

and

$$g(N, \xi) = 1. \tag{2.8}$$

In case we take another coordinate neighborhood $\mathcal{U}^* \subset M$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \phi$, denote by N^* the corresponding vector field given by (2.6). Then $\xi^* = \alpha \xi$ and $N^* = (1/\alpha)N$. Hence we have a line vector bundle over M whose sections satisfy (2.7) and (2.8). Finally, it is easy to check that any line bundle satisfying (2.7) and (2.8) has sections given by (2.6). Clearly N is nowhere tangent to M. Therefore we may state the following result on which is based the theory of degenerate hypersurfaces.

Theorem 2.2. *Let $(M, g, S(TM))$ be a degenerate hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a unique line vector bundle $tr(TM)$ over M such that its sections satisfy (2.7) and (2.8) on any coordinate neighborhood.*

By using (1.1), (1.2) and Theorem 2.2 we obtain

$$T\bar{M}_{|M} = S(TM) \perp (TM \oplus tr(TM)) = TM \oplus tr(TM). \tag{2.9}$$

The last decomposition in (2.9) is a motivation for us to call $tr(TM)$ the *transversal vector bundle* of M with respect to $S(TM)$. For the construction of the transversal vector bundle to a degenerate submanifold of arbitrary codimension see [4].

Example 2.1. In R^4_1 consider the null cone M given by the equation

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0, \quad x^0 \neq 0.$$

It is easy to verify that

$$\xi = \sum_{A=0}^{3} x^A \frac{\partial}{\partial x^A},$$

is a null vector field tangent to M and $\bar{g}(\xi, X) = 0$ for any $X \in \Gamma(TM)$. Hence $TM^\perp = Span\ \{\xi\}$. Then consider along M the vector field

$$N = \frac{1}{2(x^0)^2}\left(-x^0\frac{\partial}{\partial x^0} + \sum_{a=1}^{3} x^a \frac{\partial}{\partial x^a}\right).$$

By using $\bar{g}$ from (1.4) in case of R^4_1 we deduce $\bar{g}(N, \xi) = 1$ and $\bar{g}(N, N) = 0$. Hence we may take $tr(TM) = Span\ \{N\}$ and obtain

$$S(TM) = \left\{X \in \Gamma(TM) :\ X = \sum_{a=1}^{3} X^a \frac{\partial}{\partial x^a}, \quad \sum_{a=1}^{3} x^a X^a = 0\right\}.$$

In a similar way it follows that the null cone of R^{m+2}_q is a degenerate hypersurface. In this case we have

$$\xi = \sum_{A=0}^{m+1} x^A \frac{\partial}{\partial x^A},$$

$$N = \frac{1}{2\sum_{i=0}^{q-1}(x^i)^2}\left\{-\sum_{i=0}^{q-1} x^i \frac{\partial}{\partial x^i} + \sum_{a=q}^{m+1} x^a \frac{\partial}{\partial x^a}\right\},$$

$$S(TM) = \left\{X \in \Gamma(TM) : X = \sum_{A=0}^{m+1} x^A \frac{\partial}{\partial x^A}, \sum_{i=0}^{q-1} x^i X^i = 0, \sum_{a=q}^{m+1} x^a X^a = 0\right\}.$$

Example 2.2. In R^4_2 consider the hypersurface M given by the equation

$$x^3 = x^0 + \frac{1}{2}(x^1 + x^2)^2.$$

It is easy to verify that M is a degenerate hypersurface and

$$TM^\perp = Span\left\{\xi = \frac{\partial}{\partial x^0} + (x^1 + x^2)\frac{\partial}{\partial x^1} - (x^1 + x^2)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\right\}.$$

Consider

$$S(TM) = Span\left\{W_1 = \frac{\partial}{\partial x^1} - (x^1 + x^2)\frac{\partial}{\partial x^0}, W_2 = \frac{\partial}{\partial x^2} + (x^1 + x^2)\frac{\partial}{\partial x^3}\right\}$$

and obtain

$$N = \frac{1}{2(1+(x^1+x^2)^2)}\left\{\frac{\partial}{\partial x^0} + (x^1+x^2)\frac{\partial}{\partial x^1} + (x^1+x^2)\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}\right\}.$$

Example 2.3. We cut the unit pseudosphere $S_1^{2m+1}(1)$ by the hyperplane $x^0 - x^1 = 0$ and obtain a degenerate hypersurface M of $S_1^{2m+1}(1)$ with

$$TM^\perp = Span\left\{\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right\}.$$

Consider the screen distribution $S(TM)$ spanned by

$$\left\{W_{i-1} = x^{2m+1}\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^{2m+1}}\right\}, \quad i \in \{2, \ldots, 2m\}.$$

Then by using (2.6) we obtain the transversal vector bundle spanned by

$$N = -\frac{1}{2}\left\{(1+(x^0)^2)\frac{\partial}{\partial x^0} + ((x^0)^2-1)\frac{\partial}{\partial x^1} + 2x^0\sum_{a=2}^{2m+1} x^a\frac{\partial}{\partial x^a}\right\}.$$

Now, we come back to the general study of a degenerate hypersurface (M, g) of $(\bar{M}, \bar{g})$. Denote by $\bar{\nabla}$ the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then by using the second decomposition in (2.9) we obtain

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \text{and} \tag{2.10}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$. It follows that ∇ is a torsion-free linear connection on M, but in general g is not parallel with respect to ∇. More precisely, by using (2.10) and taking into account that $\bar{\nabla}$ is a metric connection, we obtain

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \quad \forall\, X, Y, Z \in \Gamma(TM), \tag{2.12}$$

where

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM). \tag{2.13}$$

B is a symmetric bilinear form on $\Gamma(TM)$ which we call the *second fundamental form* of M. The following result is important for the study which follows in the paper.

Proposition 2.1. The second fundamental form of M does not depend on the screen distribution on M.

Proof. Suppose $S(TM)$ and $S(TM)'$ are two screen distributions on M and B and B' are the corresponding second fundamental forms. Then by using (2.10) and (2.8) for both distributions we obtain

$$B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi) = B'(X,Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.14}$$

which proves our assertion.

As a consequence of (2.14) we deduce

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM); \tag{2.15}$$

that is, B is degenerate.

A_N from (2.11) is a $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$ which we call the *shape operator* of M. The main property of A_N is stated in the next proposition.

Proposition 2.2. The shape operator of a degenerate hypersurface has an eigenvalue $\lambda = 0$.

Proof. From (2.11), taking into account that N is a null vector field, we obtain

$$\bar{g}(A_N X, N) = 0 \quad \forall\, X \in \Gamma(TM). \tag{2.16}$$

Thus $A_N X$ has no component in $TM^\perp$ which implies rank $A_N < m+1$, and we get the assertion.

Remark 2.1. In general, the 1-form τ from (2.11) is not identically zero as it is in case of non-degenerate hypersurfaces. For the null cone we have $\tau(\xi) = -1$ (see Example 4.3).

Remark 2.2. We note that both B and τ depend on the section ξ of $TM^\perp$. Indeed, in case we have $\xi^* = \alpha\xi$, it follows $N^* = (1/\alpha)N$ and from (2.10) and (2.11) we get $B^* = \alpha B$ and $\tau(X) = \tau^*(X) + X(\log \alpha)$ for any $X \in \Gamma(TM)$.

As in the case of non-degenerate hypersurfaces we call (2.10) and (2.11) the *Gauss* and *Weingarten formulas* for the degenerate hypersurface M.

Now, according to the decomposition (2.4), we set locally

$$\nabla_X Y = \nabla^*_X Y + C(X,Y)\xi, \tag{2.17}$$

and

$$\nabla_X \xi = -A^*_\xi X + \mathcal{E}(X)\xi, \tag{2.18}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(S(TM))$. We call C the *second fundamental form* of $S(TM)$. From (2.17) we deduce that C is symmetric on $\Gamma(S(TM))$ if and only if $S(TM)$ is integrable. A^*_ξ is a $\Gamma(S(TM))$-valued linear operator on $\Gamma(TM)$ and we call it the *shape operator* of $S(TM)$. By using (2.18), (2.10) and (2.15) we obtain $\mathcal{E} = -\tau$. Hence (2.18) becomes

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi. \tag{2.19}$$

We have to note that B and A_N are not related as in the case of non-degenerate hypersurfaces. More precisely, by using (2.10), (2.11), (2.17) and (2.19) we easy obtain

$$B(X,Y) = \bar{g}(A^*_\xi X, Y), \quad \forall\, X, Y \in \Gamma(TM), \tag{2.20}$$

and

$$C(X,Z) = \bar{g}(A_N X, Z), \quad \forall\, X \in \Gamma(TM), \quad Z \in \Gamma(S(TM)). \tag{2.21}$$

Take $X = \xi$ in (2.20) and by using (2.15) we derive

$$A^*_\xi \xi = 0. \tag{2.22}$$

We call (2.17) and (2.18) the *Gauss* and *Weingarten formulas* for the screen distribution $S(TM)$.

We show here how to construct a screen distribution on a degenerate hypersurface of a time-orientable Lorentz manifold. To this end we recall (see [19], p. 149) that on a time-orientable Lorentz manifold $(\bar{M}, \bar{g})$ there exists a unit timelike vector field which we denote by L. Denote by D the timelike distribution spanned by L on $\bar{M}$ and set

$$T\bar{M} = D \perp D^{\perp},$$

where $D^{\perp}$ is the spacelike distribution that is complementary orthogonal to D. Thus at any point of M, ξ has the unique decomposition

$$\xi = \xi^{-} + \xi^{+}, \tag{2.23}$$

where $\xi^{-} \in \Gamma(D)$ and $\xi^{+} \in \Gamma(D^{\perp})$. Suppose $\xi^{-} = aL$ and define

$$V = -aL, \tag{2.24}$$

where a is a differentiable function on $\mathcal{U} \subset M$. It is easy to check that V is nowhere tangent to M. Indeed $\bar{g}(V, \xi) = a^2 \neq 0$ at any point of M, otherwise $\xi^{-} = 0$ which together with (2.23) implies $\xi = 0$, and this is a contradiction. As L is globally defined on $\bar{M}$ we obtain a line bundle $Span\{V\}$ over M. We consider the vector bundle $K = TM^{\perp} \oplus Span\{V\}$ and claim that it is non-degenerate. In fact, suppose there exists a point $x \in M$ and a vector $X_x \in K_x$ such that

$$\bar{g}(X_x, \xi_x) = 0; \quad \bar{g}(X_x, V_x) = 0.$$

From the first equality it follows $X_x \in T_xM$. As $T_xM \cap K_x = T_xM^{\perp}$ we deduce that X_x is colinear with ξ_x, and hence $\bar{g}(V_x, X_x) \neq 0$, which contradicts the above second equality. Finally, denote by $S(TM)$ the orthogonal complementary vector bundle to K in $T\bar{M}_{|M}$. As $S(TM)$ is ortogonal to $TM^{\perp}$, $S(TM) \cap TM^{\perp} = \{0\}$ and it is of rank m, it follows that $S(TM)$ is a distribution on M and $TM = S(TM) \perp TM^{\perp}$. Hence $S(TM)$ is a screen distribution on M which from now on we call the *canonical screen distribution* on M. By using (2.6) and (2.24) we obtain

$$N = \frac{1}{a^2}\left(V + \frac{1}{2}\xi\right). \tag{2.25}$$

We call the vector bundle spanned by N the *canonical transversal bundle* of M.

Theorem 2.3. *Let M be a time-orientable Lorentz manifold such that the timelike distribution D is parallel with respect to $\bar{\nabla}$. Then the canonical screen distribution on M is integrable.*

Proof. Let $X, Y \in \Gamma(S(TM))$. Then by using (2.25) and taking into account that $\bar{\nabla}$ is a metric connection we derive

$$\begin{aligned} \bar{g}([X,Y],N) &= -\frac{1}{a}\bar{g}([X,Y],L) = -\frac{1}{a}\bar{g}(\bar{\nabla}_X Y - \bar{\nabla}_Y X, L) \\ &= \frac{1}{a}\{\bar{g}(Y,\bar{\nabla}_X L) - \bar{g}(X,\bar{\nabla}_Y L)\} = 0. \end{aligned}$$

Due to (2.8) we conclude that $[X,Y]$ has no component with respect to ξ. Hence $[X,Y] \in \Gamma(S(TM))$, that is, $S(TM)$ is integrable.

The above canonical screen distribution has been constructed (see [3]) for any degenerate hypersurface M of R_q^{m+2}, $q > 1$, in the following way. Suppose M is locally given by the equations

$$x^A = f^A(u^0,\ldots,u^m), \quad A \in \{0,\ldots,m+1\}.$$

Then $TM^{\perp}$ is spanned by

$$\xi = \sum_{i=0}^{q-1}(-1)^i D^i \frac{\partial}{\partial x^i} + \sum_{a=q}^{m+1}(-1)^{a-1} D^a \frac{\partial}{\partial x^a},$$

where D^A are the determinants

$$D^A = \begin{vmatrix} \dfrac{\partial f^0}{\partial u^0} & \cdots & \dfrac{\partial f^{A-1}}{\partial u^0} & \dfrac{\partial f^{A+1}}{\partial u^0} & \cdots & \dfrac{\partial f^{m+1}}{\partial u^0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f^0}{\partial u^m} & \cdots & \dfrac{\partial f^{A-1}}{\partial u^m} & \dfrac{\partial f^{A+1}}{\partial u^m} & \cdots & \dfrac{\partial f^{m+1}}{\partial u^m} \end{vmatrix}.$$

Then locally on $\mathcal{U} \subset M$ we consider the vector field

$$V = \sum_{i=0}^{q-1}(-1)^{i-1} D^i \frac{\partial}{\partial x^i}, \tag{2.26}$$

which is nowhere tangent to M. Moreover, we show that all vector fields V span a line bundle H over M. Then it is proved that the complementary orthogonal vector bundle to the non-degenerate vector bundle $K = H \oplus TM^{\perp}$ is a screen distribution on M. This is the canonical distribution on M.

Remark 2.3. In general, the canonical screen distribution on a degenerate hypersurface of R_q^{m+2} with $q > 1$ is not integrable. In R_2^4 consider M from Example 2.2 and conclude that the canonical screen distribution is spanned by $\{W_1, W_2\}$. But $[W_1, W_2] = \partial/\partial x^0 + \partial/\partial x^3$, which does not lie in $\Gamma(S(TM))$.

3. THE GAUSS-CODAZZI EQUATIONS AND THE FUNDAMENTAL THEOREM FOR DEGENERATE HYPERSURFACES

Let M be a degenerate hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Denote by R and $\bar{R}$ the curvature tensor fields of ∇ and $\bar{\nabla}$ respectively. Then by using (2.10) and (2.11) we obtain

$$\bar{R}(X,Y)Z = R(X,Y)Z + B(X,Z)A_NY - B(Y,Z)A_NX + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N, \tag{3.1}$$

for any $X, Y, Z \in \Gamma(TM)$, where we set

$$(\nabla_X B)(Y,Z) = X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{3.2}$$

Taking the components of $\bar{R}(X,Y)Z$ with respect to $S(TM)$, $TM^{\perp}$ and $tr(TM)$, by direct calculations we obtain the following result.

Theorem 3.1. (cf. [6]). *Let $(M, g, S(TM))$ be a degenerate hypersurface of the semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the Gauss- Codazzi equations of M are given by:*

$$\bar{g}(\bar{R}(X,Y)Z, W) = g((R(X,Y)Z, W) + B(X,Z)C(Y,W) - B(Y,Z)C(X,W), \tag{3.3}$$

$$\bar{g}(\bar{R}(X,Y)Z, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \tag{3.4}$$

$$(3.5)\qquad \begin{aligned} \bar{g}(\bar{R}(X,Y)W,N) &= \bar{g}(R(X,Y)W,N) \\ &= (\nabla_X C)(Y,W) - (\nabla_Y C)(X,W) \\ &\quad +\tau(Y)C(X,W) - \tau(X)C(Y,W), \end{aligned}$$

$$(3.6)\ \begin{aligned} \bar{g}(\bar{R}(X,Y)\xi,N) &= \bar{g}(R(X,Y)\xi,N) = C(Y,A^*_\xi X) - C(X,A^*_\xi Y) \\ &\quad -2d\tau(X,Y), \end{aligned}$$

for any $X,Y,Z \in \Gamma(TM)$ and $W \in \Gamma(S(TM))$, where we set

$$(3.7)\ (\nabla_X C)(Y,W) = X(C(Y,W)) - C(\nabla_X Y,W) - C(Y,\nabla^*_X W),$$

and

$$(3.8)\qquad d\tau(X,Y) = \frac{1}{2}\{X(\tau(Y)) - Y(\tau(X)) - \tau([X,Y])\}.$$

As in the case of Riemannian manifolds we define the *Ricci tensor* of the induced connection ∇ on $(M,g,S(TM))$ by

$$Ric\,(X,Y) = \text{trace}\,\{Z \rightarrow R(X,Z)Y\},\quad \forall\, X,Y \in \Gamma(TM).$$

Consider the local frame field $\{W_i,\xi\}$ on $\mathcal{U} \subset M$, where $\{W_i\}$, $i \in \{1,\ldots,m\}$ is an orthonormal basis of $\Gamma(S(TM)_{|\mathcal{U}})$. Then we deduce

$$(3.9)\qquad Ric\,(X,Y) = \sum_{i=1}^{m} \varepsilon_i g(R(X,W_i)Y,W_i) + \bar{g}(R(x,\xi)Y,N),$$

where $\varepsilon_i = -1$ or $+1$ according as W_i is timelike or spacelike respectively. By using the Bianchi first identity with respect to ∇ and taking account of (3.3) in (3.9) we obtain

$$(3.10)\qquad \begin{aligned} Ric\,(X,Y) - Ric\,(Y,X) &= \sum_{i=1}^{m} \varepsilon_i\{C(X,W_i)B(Y,W_i) \\ &\quad -C(Y,W_i)B(X,W_i)\} \\ &\quad +\bar{g}(R(X,Y)\xi,N). \end{aligned}$$

On the other hand, by direct calculation, using (2.20) and (2.21), we deduce

$$(3.11)\qquad C(Y,A^*_\xi X) = \sum_{i=1}^{m} \varepsilon_i B(X,W_i)C(Y,W_i).$$

Finally, by using (3.11) and (3.6) in (3.10) we obtain

$$Ric\,(X,Y) - Ric(Y,X) = -2d\tau(X,Y), \quad \forall X,Y \in \Gamma(TM).$$

Hence we may state the following important result.

Theorem 3.2. *Let $(M, g, S(TM))$ be a degenerate hypersurface of $(\bar{M}, \bar{g})$. Then the Ricci tensor of the induced connection ∇ is symmetric if and only if on each $\mathcal{U} \subset M$ there exists a closed 1- form τ, i.e., $d\tau = 0$ on $\mathcal{U}$.*

In particular, for a degenerate hypersurface of a 4-dimensional Lorentz manifold, Theorem 3.2 was proved in [15].

Next, we want to present a Fundamental Theorem for degenerate hypersurfaces, that is, to find geometrical conditions for the existence of an immersion of a degenerate manifold in a semi-Euclidean space. To this end we start with a 1-degenerate $(m+1)$-dimensional manifold M of index $q-1$, $m > 0$, $q > 0$. Suppose there exists a line vector bundle F over M such that $E = TM \oplus F$ is a semi- Riemannian vector bundle with a semi-Riemannian metric $\bar{g}$ satisfying the condition

$$(\mathrm{C}_1) \quad \bar{g}(X,Y) = g(X,Y),\ \bar{g}(Z,V) = \bar{g}(V,V') = 0,$$

for any $X, Y \in \Gamma(TM)$, $Z \in \Gamma(S(TM))$ and $V, V' \in \Gamma(F)$. Since $\bar{g}$ is non-degenerate on E we have $\bar{g}(U,V) \neq 0$ for any non-zero vector fields $U \in \Gamma(Rad\,TM)$ and $V \in \Gamma(F)$.

Furthermore, we suppose there exists a torsion-free linear connection ∇' on M and a linear connection ∇'' on vector bundle F, satisfying

$$(\mathrm{C}_2) \quad \bar{g}(\nabla'_X U, V) + \bar{g}(U, \nabla''_X V) = X(\bar{g}(U,V)),$$

for any $X \in \Gamma(TM)$, $U \in \Gamma(Rad\,TM)$ and $V \in \Gamma(F)$. Consider a screen distribution $S(TM)$ on M, i.e., we have

$$TM = S(TM) \perp Rad\,TM. \tag{3.12}$$

Hence we may set

$$\nabla'_X W = \overset{*}{\nabla}{}'_X W + \overset{*}{h}{}'(X,W),\ \forall X \in \Gamma(TM),\ W \in \Gamma(S(TM)), \tag{3.13}$$

and

$$(3.14)\quad \nabla'_X U = -\overset{*}{A'}(U,X) + \overset{*}{\nabla}{}''_X U,\ \forall\, X \in \Gamma(TM),\ U \in \Gamma(Rad\, TM),$$

where $\overset{*}{\nabla}{}'_X W$ and $\overset{*}{A'}(U,X)$ belong to $\Gamma(S(TM))$ while $\overset{*}{h'}(X,W)$ and $\overset{*}{\nabla}{}''_X U$ belong to $\Gamma(Rad\ TM)$. It is easy to verify that $\overset{*}{\nabla}{}'$ and $\overset{*}{\nabla}{}''$ are linear connections on vector bundles $S(TM)$ and $Rad\, TM$ respectively, and $\overset{*}{h'}$ and $\overset{*}{A'}$ are $\mathcal{F}(M)$-bilinear forms on $\Gamma(TM) \times \Gamma(S(TM))$ and on $\Gamma(Rad\ TM) \times \Gamma(TM)$, respectively. With respect to these geometric objects we suppose the following conditions are satisfied:

$$(C_3)\quad \overset{*}{A'}(U,U) = 0,\ g(\overset{*}{A'}(U,X),Y) = g(\overset{*}{A'}(U,Y),X),$$

$$(C_4)\quad (\nabla'_X g)(W,W') = (\nabla'_X g)(U,U) = 0,\ (\nabla'_X g)(W,U) = g(\overset{*}{A'}(U,X),W)$$

$$(C_5)\quad (\nabla'_X \overset{*}{A'})(U,Y) = (\nabla'_Y \overset{*}{A'})(U,X),$$

for any $X, Y \in \Gamma(TM)$, $W, W' \in \Gamma(S(TM))$ and $U \in \Gamma(Rad\, TM)$, where we put

$$(\nabla'_X \overset{*}{A'})(U,Y) = \overset{*}{\nabla}{}'_X (\overset{*}{A'}(U,Y)) - \overset{*}{A'}(\overset{*}{\nabla}{}''_X U, Y) - \overset{*}{A'}(U, \nabla'_X Y).$$

Finally, denote by R' the curvature tensor of ∇' and suppose the following conditions are fulfilled:

$$(C_6)\quad g(R'(X,Y)Z,W) = g(\overset{*}{A'}(\overset{*}{h'}(X,W),Y) - \overset{*}{A'}(\overset{*}{h'}(Y,W),X),Z),$$

$$(C_7)\quad \bar{g}(R'(X,Y)Z,V) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$, $W \in \Gamma(S(TM))$ and $V \in \Gamma(F)$. In what follows we denote by $\hat{g}$ the semi-Euclidean metric on R^{m+2}_q given by (1.4).

Theorem 3.3 (Fundamental Theorem for Degenerate Hypersurfaces). *Let $(M, g, S(TM))$ be a 1-degenerate simply connected $(m+1)$-dimensional manifold of index $q-1$, endowed with the vector bundle F and geometric objects $\bar{g}, \nabla', \nabla'', \overset{*}{h}{}'$ and, $\overset{*}{A'}$ satisfying conditions (C_1) -*

(C_7). Then there exists a degenerate isometric immersion

$$f : (M, g, S(TM)) \to (R_q^{m+2}, \hat{g}),$$

$$i.e., \ \hat{g}(f_*X, f_*Y) = g(X, Y), \ \forall \, X, Y \in \Gamma(TM)$$

and a vector bundle isomorphism $\bar{f} : F \to tr(TfM)$, such that

$$f_*(\nabla'_X Y) = \nabla_{f_*X} f_*Y, \ \bar{f}(\nabla''_X V) = \tau(X)\bar{f}(V),$$

$$f_*(\overset{*}{A'}(U, X)) = \overset{*}{A}_{f_*U}(f_*X), \ f_*(\overset{*}{h'}(X, W)) = C(f_*X, f_*W) f_*\xi,$$

for any $X, Y \in \Gamma(TM)$, $U \in \Gamma(Rad\ TM)$, $W \in \Gamma(S(TM))$ and $V \in \Gamma(F)$, where $tr(TfM)$ is the transversal vector bundle of M with respect to $f_*(S(TM))$, and $\nabla, \tau, \overset{*}{A}$ and C are geometric objects induced on fM by the Gauss and Weingarten formulas with respect to the immersion f.

Proof. First, by using $\overset{*}{A'}$ define the $\mathcal{F}(M)$-bilinear form

(3.15) $h' : \Gamma(TM) \times \Gamma(TM) \to \Gamma(F); \bar{g}(h'(X, Y), U) = g(\overset{*}{A'}(U, X), Y),$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(Rad\ TM)$. Note that based on the last two equalities in condition (C_1), h' is well defined. Due to (C_3) we see that h' is symmetric and satisfies

$$h'(X, U) = 0, \quad \forall X \in \Gamma(TM). \tag{3.16}$$

Next, by means of $\overset{*}{h}'$, and using (3.12), we define

$A' \ : \Gamma(F) \times \Gamma(TM) \to \Gamma(TM)$ by

$$\bar{g}(A'(V, X), W) = \bar{g}(\overset{*}{h'}(X, W), V); \quad \bar{g}(A'(V, X), V') = 0, \tag{3.17}$$

for any $X \in \Gamma(TM)$, $W \in \Gamma(S(TM))$ and $V, V' \in \Gamma(F)$. These two geometric objects enable us to define the differential operator $\bar{\nabla}$ by

$$\bar{\nabla}_X Y = \nabla'_X Y + h'(X, Y), \quad \forall \, X, Y \in \Gamma(TM), \tag{3.18}$$

and

$$\bar{\nabla}_X V = -A'(V, X) + \nabla''_X V, \quad \forall \, X \in \Gamma(TM), \quad V \in \Gamma(F). \tag{3.19}$$

It is easy to verify that $\bar{\nabla}$ is a linear connection on E. Moreover, by using (3.18) we deduce

$$(3.20)\quad (\bar{\nabla}_X \bar{g})(Y,Z) = (\nabla'_X g)(Y,Z) - \bar{g}(h'(X,Y),Z) - \bar{g}(h'(X,Z),Y),$$

for any $X, Y, Z \in \Gamma(TM)$. By using (C_4), (C_1), (3.15) and (3.16) in the right hand side of (3.20) we infer

$$(\bar{\nabla}_X \bar{g})(Y,Z) = 0.$$

On the other hand, by using (C_1), (3.13) and (23.17) we obtain

$$(\bar{\nabla}_X \bar{g})(W,V) = 0, \quad \forall\, W \in \Gamma(S(TM)), \quad V \in \Gamma(F).$$

Conditions (C_1) and (C_2) yield

$$(\bar{\nabla}_X \bar{g})(U,V) = 0, \quad \forall\, U \in \Gamma(Rad\, TM), \quad V \in \Gamma(F).$$

Finally, by using (C_1), (3.19) and the second relation in (3.17) we obtain

$$(\bar{\nabla}_X \bar{g})(V,V') = 0, \quad \forall\, V, V' \in \Gamma(F).$$

Summing up, due to the above equalities, we conclude that $\bar{\nabla}$ is a metric connection on E.

Next, by using (3.15), we deduce

$$(3.21)\quad h'(X, \overset{*}{A'}(U,Y)) = h'(Y, \overset{*}{A'}(U,X)), \quad \forall\, X, Y \in \Gamma(TM), \quad U \in \Gamma(Rad\, TM).$$

Then, taking into account that $\bar{\nabla}$ is a metric connection and using (3.13) (3.15), (3.18), (319) and (3.21), we see that (C_5) is equivalent to

$$(3.22)\qquad (\nabla'_X h')(Y,Z) = (\nabla'_Y h')(X,Z), \quad \forall X, Y, Z \in \Gamma(TM),$$

where we set

$$(\nabla'_X h')(Y,Z) = \nabla''_X(h'(Y,Z)) - h'(\nabla'_X Y, Z) - h'(Y, \nabla'_X Z).$$

Moreover, by using (3.15), we see that (C_6) is equivalent to

$$g(R'(X,Y)Z,W) = \bar{g}(h'(Y,Z),\overset{*}{h'}(X,W)) - \bar{g}(h'(X,Z),\overset{*}{h'}(Y,W)), \tag{3.23}$$

By using (3.18), (3.19) , (C_7), (3.22) and (3.23) and performing some calculations similar to those for the Gauss-Codazzi equations we conclude that the curvature tensor $\bar{R}$ of $\bar{\nabla}$ vanishes identically. Now, consider a point $u \in M$ and orthonormal vectors $\{\overset{\circ}{W}_0,\ldots,\overset{\circ}{W}_{m+1}\}$ from the fibre E_u such that $\{\overset{\circ}{W}_0,\ldots,\overset{\circ}{W}_{q-1}\}$ and $\{\overset{\circ}{W}_q,\ldots,\overset{\circ}{W}_{m+1}\}$ are timelike and spacelike, respectively. Since M is simply connected and $\bar{R}$ vanishes, there exist unique global extensions $\{\overset{\circ}{W}_0,\ldots,\overset{\circ}{W}_{m+1}\}$ parallel with respect to $\bar{\nabla}$. These global sections are pointwise orthonormal and have the same causal character as $\{\overset{\circ}{W}_0,\ldots,\overset{\circ}{W}_{m+1}\}$ since $\bar{g}$ is parallel with respect to $\bar{\nabla}$.

In the present proof we use the range of indices:$A,B,\ldots \in \{0,\ldots,m+1\}$; $\alpha,\beta,\ldots \in \{0,\ldots,m\}$; $i,j,\ldots \in \{0,\ldots,q-1\}$; $a,b\ldots \in \{q,\ldots,m+1\}$.

Now we consider a coordinate system $(\mathcal{U};u^0,\ldots,u^m)$ around $u \in M$ and set $\partial/\partial u^\alpha = S^A_\alpha W_A$. Hence the local components of the degenerate metric g on M are given by

$$g_{\alpha\beta} = g\left(\frac{\partial}{\partial u^\alpha},\frac{\partial}{\partial u^\beta}\right) = -\sum_{i=0}^{q-1} S^i_\alpha S^i_\beta + \sum_{a=q}^{m+1} S^a_\alpha S^a_\beta. \tag{3.24}$$

Taking into account that $\{W_A\}$ are parallel with respect to $\bar{\nabla}$ we obtain

$$\bar{\nabla}_{\frac{\partial}{\partial u^\beta}} \frac{\partial}{\partial u^\alpha} = \frac{\partial S^A_\alpha}{\partial u^\beta} W_A. \tag{3.25}$$

By using (3.18) and (3.25) we derive

$$\frac{\partial S^A_\alpha}{\partial u^\beta} = \frac{\partial S^A_\beta}{\partial u^\alpha}. \tag{3.26}$$

Thus the 1-forms $\omega^A = S^A_\alpha du^\alpha$ are closed and therefore they are exact on $\mathcal{U}$, that is, there exist smooth functions f^A such that $\omega^A = df^A$ or, equivalently,

$$\frac{\partial f^A}{\partial u^\alpha} = S^A_\alpha.$$

Define $f : \mathcal{U} \to R^{m+2}_q$ by $f = (f^0, \ldots, f^{m+1})$ and note that

$$f_*\left(\frac{\partial}{\partial u^\alpha}\right) = (S^0_\alpha, \ldots, S^{m+1}_\alpha).$$

Then, by using (3.24), we deduce

$$\hat{g}\left(f_*\left(\frac{\partial}{\partial u^\alpha}\right), f_*\left(\frac{\partial}{\partial u^\beta}\right)\right) = g_{\alpha\beta}.$$

That is, f is a degenerate immersion of $\mathcal{U}$ in R^{m+2}_q. As a consequence, $\bar{\mathcal{U}} = f(\mathcal{U})$ becomes a degenerate hypersurface of R^{m+2}_q.

Next, define the isomorphism of vector bundles (linear isometry between fibres)

$$\Phi : T\mathcal{U} \oplus F_{|\mathcal{U}} \to TR^{m+2}_q{}_{|\bar{\mathcal{U}}}, \quad \Phi(W_A) = E_A,$$

where $\{E_A\}$ is the canonical orthonormal frame field on $\bar{\mathcal{U}}$. Note that Φ carries isometrically $T\mathcal{U}$ onto $T\bar{\mathcal{U}}$. Indeed

$$\Phi\left(\frac{\partial}{\partial u^\alpha}\right) = S^A_\alpha \Phi(W_A) = S^A_\alpha E_A = f_*\left(\frac{\partial}{\partial u^\alpha}\right).$$

Moreover, the radical distribution and the screen distribution are preserved by Φ.

The Levi-Civita connection on R^{m+2}_q with respect to $\hat{g}$ is denoted by $\hat{\nabla}$. Then taking into account that $\{W_A\}$ and $\{E_A\}$ are parallel with respect to $\bar{\nabla}$ and $\hat{\nabla}$ respectively, we infer

$$\Phi(\bar{\nabla}_X Y) = \hat{\nabla}_{f_*X} f_*Y \quad \text{and} \quad \Phi(\bar{\nabla}_X V) = \hat{\nabla}_{f_*X} \Phi V,$$

for any $X, Y \in \Gamma(T\mathcal{U})$ and $V \in \Gamma(F_{|\mathcal{U}})$. By using (3.18), (3.19) and the Gauss and Weingarten formulas for the degenerate hypersurface $\mathcal{U}$ of R^{m+2}_q, we deduce

$$f_*(\nabla'_X Y) = \nabla_{f_*X} f_*Y, \Phi(h'(X, Y)) = B(f_*X, f_*Y)\Phi V,$$

and

$$f_*(A'(V,X)) = A_{\Phi V} f_* A, \Phi(\nabla''_X V) = \tau(X)\Phi V.$$

Moreover, from (3.13) and (3.14), we obtain

$$f_*(\overset{*}{A'}(U,X)) = \overset{*}{A}_{f_*U}\, f_*X,\ \ f_*(\overset{*}{h'}(X,W)) = C(f_*X, f_*W) f_*\xi.$$

It is easy to check that all these local immersions are determined up to an isometry of R_q^{m+2}.

Now, denote by $Tr(T\bar{\mathcal{U}})$ the transversal vector bundle of $\bar{\mathcal{U}}$ with respect to the screen distribution $S(T\bar{\mathcal{U}}) = f_*(s(T\mathcal{U}))$. It follows that $tr(T\bar{\mathcal{U}}) = \Phi(F_{|\mathcal{U}})$. Thus we have a vector bundle isomorphism $\bar{f}_{|\mathcal{U}}$: $F_{|\mathcal{U}} \to tr(T\bar{\mathcal{U}})$ which is the restriction of Φ to $F_{|\mathcal{U}}$.

Therefore we constructed both f and $\bar{f}$ on $\mathcal{U} \subset M$ satisfying the relations in the theorem.

Finally, since M is simply connected, the local degenerate immersions will be glued together as in the case of non-degenerate hypersurfaces, and give us the global degenerate isometric immersion $f : M \to R_q^{m+2}$. Moreover, $S(T\mathcal{U})$ and $S(T\mathcal{U}^*)$ coincide on $\mathcal{U} \cap \mathcal{U}^*$ since they are restrictions of $S(TM)$ to coordinate neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ respectively. Hence $S(T\bar{\mathcal{U}})$ and $S(T\bar{\mathcal{U}}^*)$ coincide on $\bar{\mathcal{U}} \cap \bar{\mathcal{U}}^*$ and thus we obtain a screen distribution $S(TfM)$ of fM. Since $tr(TfM)$ is unique and coincides locally with $tr(T\bar{\mathcal{U}})$, there exists a global isomorphism $f : F \to tr(TfM)$ which, together with f, satisfy the relations of the theorem. This completes the proof of the theorem.

4. SPECIAL CLASSES OF DEGENERATE HYPERSURFACES

It is the purpose of this section to introduce and study some important classes of degenerate hypersurfaces and give examples.

Let $(M, g, S(TM))$ be a degenerate hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. If any geodesic of M with respect to an induced connection ∇ is a geodesic of $\bar{M}$ with respect to the Levi-Civita connection

$\bar{\nabla}$, we say that M is *totally geodesic*. The theorem which follows shows that the definition does not depend on the screen distribution.

Theorem 4.1. *Let $(M, g, S(TM))$ be a degenerate hypersurface of $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:*

(i) M is totally geodesic.

(ii) The second fundamental form of M vanishes identically on M.

(iii) The shape operator of $S(TM)$ vanishes identically on M.

(iv) There exists a unique torsion-free metric connection ∇ induced by $\bar{\nabla}$.

(v) $TM^{\perp}$ is a parallel distribution with respect to ∇.

(vi) $TM^{\perp}$ is a Killing distribution on M.

(vii) M is a Reinhart degenerate manifold.

The equivalence of conditions (i) through (vi) is proved in [6], and due to Theorem 1.2 we see that (vii) is equivalent with (vi).

Example 4.1. For the sake of simplicity of calculations we consider $m = 1$ in Example 2.3. Hence the degenerate surface is obtained by cuting $S_1^3(1)$ with the plane $x^0 - x^1 = 0$. Denote by ∇' and $\bar{\nabla}$ the Levi-Civita connections on R_1^4 and $S_1^3(1)$ respectively. Then we have

$$\nabla'_X Y = \bar{\nabla}_X Y + h(X,Y)N' = \nabla_X Y + B(X,Y)N + h(X,Y)N',$$

for any $X, Y \in \Gamma(TM)$, where h is the second fundamental form of $S_1^3(1)$ in R_1^4 and N' is the position vector field on $S_1^3(1)$. Since $B(X,\xi) = 0$ for any $X \in \Gamma(TM)$ we only need to calculate $B(W_1, W_1)$, where $W_1 = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}$. Thus we set

$$\begin{aligned}\nabla'_{W_1} W_1 \;&=\; \alpha\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right) + \beta\left(x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3}\right) \\ &\quad -\frac{1}{2}B(W_1,W_1)\left\{(1+(x^0)^2)\frac{\partial}{\partial x^0} + ((x^0)^2-1)\frac{\partial}{\partial x^1}\right.\end{aligned}$$

$$+2x^0x^2\frac{\partial}{\partial x^2}+2x^0x^3\frac{\partial}{\partial x^3}\bigg\}+h(W_1,W_1)\sum_{A=0}^{3}x^A\frac{\partial}{\partial x^A}.$$

Using (1.5) for ∇' we obtain $\alpha = x^0$, $\beta = 0$, $B(W_1, W_1) = 0$ and $h(W_1, W_1) = -1$. Thus M is a totally geodesic degenerate surface of $S_1^3(1)$ and the unique induced metric connection on M is given by

$$\nabla_{W_1}W_1 = x^0\xi, \quad \nabla_{W_1}\xi = \nabla_\xi W_1 = 0.$$

In a similar way can be proved that any degenerate great hypersphere of $S_1^m(r)$ is totally geodesic.

Example 4.2. A hyperplane $M : x^0 = c + c_1x^1 + \ldots + c_{m+1}x^{m+1}$ is degenerate in R_q^{m+2} if and only if

$$1+\sum_{i=1}^{q-1}(c_i)^2 = \sum_{a=q}^{m+1}(c_a)^2.$$

In this case $TM^\perp$ is spanned by

$$\xi = \frac{\partial}{\partial x^0} - \sum_{i=1}^{q-1}c_i\frac{\partial}{\partial x^i} + \sum_{a=q}^{m+1}c_a\frac{\partial}{\partial x^a},$$

and therefore $\bar{\nabla}_X\xi = 0$ for any $X \in \Gamma(TM)$, where $\bar{\nabla}$ is the Levi-Civita connection on R_q^{m+2}. Hence M is totally geodesic in R_q^{m+2}.

Next, we say that a degenerate hypersurface M of $(\bar{M}, \bar{g})$ is *totally umbilical* if there exists locally on each $\mathcal{U} \subset M$ a function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall\, X, Y \in \Gamma(TM).$$

According to Proposition 2.1 the definition does not depend on the screen distribution. Therefore, due to (2.15), M is totally umbilical if and only if there exists a function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall\, X, Y \in \Gamma(S(TM)), \tag{4.1}$$

where $S(TM)$ is a screen distribution on M. By using (2.20) and (4.1) it follows that M is totally umbilical if and only if the shape operator of a screen distribution $S(TM)$ satisfies

$$\overset{*}{A}_\xi X = \rho X, \quad ,\forall\, X \in \Gamma(S(TM)). \tag{4.2}$$

Theorem 4.2 ([6]). *Let $(M, g, S(TM))$ be a totally umbilical degenerate hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold of constant curvature $(\bar{M}(c), \bar{g})$. then ρ satisfies the partial differential equation*

$$\xi(\rho) + \rho\tau(\xi) - \rho^2 = 0,$$

and the curvature tensor of M is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} + \rho\{g(Y,Z)A_N X - g(X,Z)A_N Y\},$$

for any $X, Y, Z \in \Gamma(TM_{|\mathcal{U}})$. Moreover, if $m > 1$ then ρ satisfies the partial differential equations

$$X(\rho) + \rho\tau(X) = 0, \quad \forall\, X \in \Gamma(S(T\mathcal{U})_{|\mathcal{U}}).$$

The above necessary conditions for M to be totally umbilical seems to be very strong conditions on the geometry of M. That is why we need to prove the existence of totally umbilical submanifolds.

Example 4.3. Suppose M is the null cone of R^{m+2}_q. Then, based on Example 2.1, ξ is the position vector field, and it is globally defined on M. Thus, by using (2.10), (2.15) and (1.5), we obtain

$$\nabla_X \xi = \bar{\nabla}_X \xi = X, \quad \forall X \in \Gamma(TM). \tag{4.3}$$

Next, by using (2.19) in (4.3), we deduce

$$A^*_\xi X + \tau(X)\xi + X = 0, \quad \forall\, X \in \Gamma(TM). \tag{4.4}$$

Hence for any $X \in \Gamma(S(TM))$ from (4.4) we derive

$$\tau(X) = 0, \tag{4.5}$$

and

$$\overset{*}{A}_\xi\, X = -X,$$

that is, M is totally umbilical and $\rho = -1$ with respect to the above ξ. Moreover, from (4.4) and (2.22) we obtain $\tau(\xi) = -1$. Hence Theorem 3.2 implies the following important result for the geometry of the null cone.

Theorem 4.3. *The Ricci tensor of the induced connection* ∇ *on the null cone of* R_q^{m+2} *is symmetric.*

By using the general theory we developed in the previous sections for degenerate hypersurfaces, we may obtain new results on the geometry of the null cone. The next theorems support this assertion.

Theorem 4.4. *On the null cone* M *of* R_q^{m+2} *there exists a foliation of codimension 1.*

Proof. We prove that the screen distribution presented in Example 2.1 is integrable. First, by using (1.5) and N from Example 2.1, we obtain

$$\bar{g}(Y, \bar{\nabla}_X N) = \frac{1}{2\sum_{i=0}^{q-1}(x^i)^2} \sum_{A=0}^{m+1} X^A Y^A, \tag{4.6}$$

where $X = X^A(\partial/\partial x^A)$ and $Y = Y^A(\partial/\partial x^A)$ belong to $\Gamma(S(TM))$. As $\bar{\nabla}$ is a torsion - free metric connection we deduce

$$\bar{g}([X,Y], N) = \bar{g}(X, \bar{\nabla}_Y N) - \bar{g}(Y, \bar{\nabla}_X N) = 0,$$

for any $X, Y \in \Gamma(S(TM))$. Hence $[X,Y] \in \Gamma(S(TM))$, and this completes the proof.

Theorem 4.5. *Let* M *be the null cone of* R_q^{m+2}*,* $m > 0$ *. Then* $A_N\xi = 0$ *and any other eigenvector field* $Y \neq \alpha\xi$ *is either spacelike or timelike according as the corresponding eigenfunction is negative or positive, respectively.*

Proof. Take $X \in \Gamma(S(TM))$. By using (1.5), (2.10) and (2.15) we obtain

$$\nabla_\xi X = \bar{\nabla}_\xi X = x^A \frac{\partial X^B}{\partial x^A} \frac{\partial}{\partial x^B}. \tag{4.7}$$

Differentiating

$$\sum_{i=0}^{q-1} x^i X^i = 0 \quad \text{and} \sum_{a=q}^{m+1} x^a X^a = 0,$$

with respect to x^j and x^b, respectively, we easily obtain

$$\sum_{i=0}^{q-1} x^i x^A \frac{\partial X^i}{\partial x^A} = 0, \quad \sum_{a=q}^{m+1} x^a x^A \frac{\partial X^a}{\partial x^A} = 0.$$

Thus, taking account of (4.7), we deduce that $\nabla_\xi X \in \Gamma(S(TM))$ and (2.17) implies

$$C(\xi, X) = 0. \tag{4.8}$$

This, combined with (2.16) and (2.21), yields $A_N\xi = 0$. Now, denote by P the projection morphism of TM on $S(TM)$ and use (2.4) to obtain

$$Y = PY + \eta(Y)\xi, \quad \forall Y \in \Gamma(TM). \tag{4.9}$$

Then by using (4.9), (2.21), (4.8), (2.11) and (4.6) we infer

$$\begin{aligned} g(A_NY, Y) &= g(A_NPY + \eta(Y)A_N\xi, PY) = g(A_NPY, PY) \\ &= -\frac{1}{2\sum_{i=0}^{q-1}(x^i)^2}\sum_{A=0}^{m+1}\{(PX)^A\}^2 < 0, \end{aligned} \tag{4.10}$$

for any $Y \neq \alpha\xi$. If Y is an eigenvector field and λ is the eigenfunction, i.e., $A_NY = \lambda Y$, we can use (4.10) to deduce

$$\lambda g(Y, Y) < 0,$$

which proves the theorem.

Furthermore, as $c = 0$ and $\rho = -1$ for the null cone M of R_q^{m+2}, , we derive from Theorem 4.2 that

$$R(X, Y)Z = g(X, Z)A_NY - g(Y, Z)A_NX, \ \forall\, X, Y, Z \in \Gamma(TM). \tag{4.11}$$

In particular, we obtain the following result.

Theorem 4.6. *Let M be the null cone of the Lorentz space R_1^{m+2}. Then we have*

(i) $A_N\xi = 0, \quad A_N = -\dfrac{1}{2(x^0)^2}X,$ *for any $X \in \Gamma(S(TM))$.*

(ii) The curvature tensor of the induced connection on M is given by

$$R(X, Y)Z = \frac{1}{2(x^0)^2}\{g(Y, Z)PX - g(X, Z)PY\}, \quad \forall X, Y, Z \in \Gamma(TM).$$

Proof. Consider $X, Y \in \Gamma(S(TM))$ and, by using (2.11) and (4.6), obtain

$$g(Y, A_N X) = -\bar{g}(Y, \bar{\nabla}_X N) = -\frac{1}{2(x^0)^2} \sum_{a=1}^{m+1} X^a Y^a = -\frac{1}{2(x^0)^2} g(X, Y),$$

since $X^0 = Y^0 = 0$. Thus by Theorem 4.5 we have the assertion (i). Finally, taking into account of (4.11) and assertion (i), we derive the formula in assertion (ii).

Theorem 4.7. *a degenerate surface M of a 3-dimensional Lorentz manifold $\bar{M}$ is either totally umbilical or totally geodesic.*

Proof. Let $\mathcal{U}$ be a coordinate neighborhood of M and $S(TM)$ be a screen distribution spanned by W on $\mathcal{U}$. If M is not totally geodesic, define $\rho = B(W, W)/g(W, W)$ and (4.1) is satisfied. Hence M is totally umbilical.

It is interesting to investigate the existence of some other totally umbilical degenerate hypersurfaces of semi-Euclidean spaces. In this respect the author succeeded in determining all totally umbilical degenerate hypersurfaces of R_2^4 (cf. [5]).

5. DEGENERATE HYPERSURFACES OF LORENTZ SPACES

Let M be a degenerate hypersurface of the Lorentz space R_1^{m+2} given by the equation

(5.1) $$F(x^0, \ldots, x^{m+1}) = 0,$$

where F is differentiable on a domain $D \subset R^{m+2}$ and rank $[F'_{x^0} \ldots F'_{x^{m+1}}] = 1$ on M. Moreover, according to (2.3), the partial derivatives of first order of F satisfy

(5.2) $$(F'_{x^0})^2 = \sum_{a=1}^{m+1} (F'_{x^a})^2.$$

Thus $F'_{x^0} \neq 0$ on M, which enables us to consider $TM^{\perp}$ spanned by

(5.3) $$\xi = -\frac{\partial}{\partial x^0} + \frac{1}{F'_{x^0}} \sum_{a=1}^{m+1} F'_{x^a} \frac{\partial}{\partial x^a}.$$

Consider the transversal vector bundle spanned by

$$N = \frac{\partial}{\partial x^0} + \frac{1}{2}\xi. \tag{5.4}$$

$X = X^A(\partial/\partial x^A)$ belongs to $\Gamma(S(TM))$ if and only if

$$X^0 = 0 \quad \text{and} \quad \sum_{a=1}^{m+1} X^a F'_{x^a} = 0. \tag{5.5}$$

Remark 5.1. As R_1^{m+2} is a time-orientable Lorentz manifolds with $L = \partial/\partial x^0$, comparing (2.25) with (5.4) we see that the above distribution is just the canonical screen distribution on M.

By using (1.5) and (5.4) we obtain

$$\bar{g}(Y, \bar{\nabla}_X N) = \frac{1}{2}\bar{g}(Y, \bar{\nabla}_X \xi) = -\frac{1}{2}g(Y, A_\xi^* X) = -\frac{1}{2}B(X, Y), \tag{5.6}$$

for any $X, Y \in \Gamma(TM)$. Since B is symmetric, by using (2.6) we derive

$$\begin{aligned} \bar{g}([X,Y], N) &= \bar{g}(\bar{\nabla}_X Y - \bar{\nabla}_Y X, N) \\ &= \bar{g}(X, \bar{\nabla}_Y N) - \bar{g}(Y, \bar{\nabla}_X N) = 0 \end{aligned} \tag{5.7}$$

and

$$\bar{g}(\nabla_\xi X, N) = \bar{g}(\bar{\nabla}_\xi X, N) = -\bar{g}(X, \bar{\nabla}_\xi N) = \frac{1}{2}B(\xi, X) = 0, \tag{5.8}$$

for any $X, Y \in \Gamma(S(TM))$. In view of (5.7) we may state the following result.

Theorem 5.1. *The canonical screen distribution of a degenerate hypersurface M of R_1^{M+2} is integrable.*

Taking account of (2.11) and (2.21) in (5.6) we see that the fundamental forms of M and $S(TM)$ are related by

$$B(X, Y) = 2C(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{5.9}$$

In a similar way, from (5.8) we deduce

$$C(\xi, X) = 0, \quad \forall\, X \in \Gamma(S(TM)). \tag{5.10}$$

Next, using (2.11) and (5.6), and taking account of (2.15), we derive

$$\tau(X) = \bar{g}(\bar{\nabla}_X N, \xi) = -\frac{1}{2} B(\xi, X) = 0. \tag{5.11}$$

Hence, due to Theorem 3.2 and (5.11) we state the following result.

Theorem 5.2. *The Ricci tensor of the induced connection on any degenerate hypersurface M of R_1^{m+2} is symmetric.*

Taking account of (5.9) - (5.11) we see that the Gauss-Codazzi equations of (3.3) - (3.6) become

$$\begin{aligned} g(R(X,Y)Z, PW) &= 2\{C(PY,PZ)C(PX,PY) \\ &\quad - C(PX,PZ)C(PY,PW)\}, && (5.12)\\ (\nabla_X B)(Y,Z) &= (\nabla_Y B)(X,Z), && (5.13)\\ \bar{g}(R(X,Y)Z, N) &= 0, && (5.14)\end{aligned}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where P is the projection morphism of TM on the canonical screen distribution. Thus, by (5.12) and (5.14), we deduce that the curvature tensor of a degenerate hypersurface of R_1^{m+2} is given by

$$R(X,Y)Z = 2\{C(PY,PZ)PX - C(PX,PZ)PY\}. \tag{5.15}$$

Now, suppose M is a totally umbilical degenerate hypersurface of R_1^{m+2}. Then by using (4.1), (5.9) and (5.15) we deduce the following result.

Theorem 5.3. *The curvature tensor of a totally umbilical degenerate hypersurface of R_1^{m+2} is given by*

$$R(X,Y)Z = \rho\{g(PY,PZ)PX - g(PX,PZ)PY\}.$$

Now, suppose M^* is a leaf of the canonical screen distribution and R^* is the curvature tensor field of the induced connection on M^* by $\bar{\nabla}$.

Then by direct calculations, using (2.17), (3.5), (5.11), (2.20) and (5.9), we obtain

$$(5.16)\quad g(R(X,Y)Z,W) = g(R^*(X,Y)Z,W) + 2\{C(X,Z)C(Y,W) - C(Y,Z)C(X,W)\},$$

for any $X, Y, Z, W \in \Gamma(TM^*)$. Comparing (5.16) with (5.15) we conclude

$$(5.17)\quad R(X,Y)Z = \frac{1}{2}R^*(PX,PY)PZ, \quad \forall\, X,Y,Z \in \Gamma(TM)$$

Finally, taking account of (2.10), (2.17) and (5.9) we deduce

$$(5.18)\quad B^*(X,Y) = B(X,Y)\left(\frac{1}{2}\xi + N\right),$$

for any $X, Y \in \Gamma(TM^*)$, where B^* is the second fundamental form of M^* as a non-degenerate submanifold of codimension two in R_1^{m+2}. Based on (5.17) and (5.18) we may state the following important result.

Theorem 5.4. *A degenerate hypersurface M of R_1^{m+2} is*

(i) flat

(ii) totally umbilical

(iii) totally geodesic

if and only if, any leaf of the canonical screen distribution is immersed as a non-degenerate submanifold of codimension two in R_1^{m+2}.

Thus, as a final conclusion we may say that the differential geometry of a degenerate hypersurface of a Lorentz space is intimately related to the geometry of an arbitrary leaf of the canonical screen distribution. In particular, if M is a degenerate Monge hypersurface of R_1^4, the results of this section are included in [7].

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