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MARTINGALE REPRESENTATION THEOREM IN INFINITE DIMENSIONS*

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ABSTRACT. In this article we prove a martingale representation theorem for Hilbert space valued martingales, adapted to filtration generated by a given Wiener process W on another separable Hilbert space H. Two cases are considered: first when W is cylindrical, and second when W is a genuine Q - Wiener process on H. A Clark-Ocone theorem is derived in this setting to give an explicit form for the integrand in this theorem.

Keywords. Cylindrical Wiener process, Wiener space, martingale representation theorem, Clark-Ocone theorem.

1. INTRODUCTION

Let H and K be two separable Hilbert spaces and W be a Q - Wiener process in H or, more generally, a cylindrical Wiener process on H.

In this article we shall provide a martingale representation theorem for martingales M in K, which are adapted to the filtration generated by the cylindrical Wiener process W, $\{\mathcal{F}_t(W)\ ,\ 0 \leq t \leq T\}$; see Section 2 for definition. This representation takes the form

$$M(t) = M(0) + \int_0^t R(s)\, dW(s),\ 0 \leq t \leq T,$$

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for some R which is progressively measurable and is Hilbert-Schmidt. We also consider the case when the process W is a $\mathcal{Q}$-Wiener process (defined in Section 3); in which case R is determined such that $R\,\mathcal{Q}^{1/2}$ is Hilbert-Schmidt. These are the results of Theorem 3.1 and Corollary 3.1 in Section 3 below.

On the other hand, to be able to determine the process R under more regularity conditions, we derive a Clark-Ocone formula using the Malliavin calculus.

This maringale representation theorem is proved to be very useful in many aspects in stochastic analysis, for instance when working in infinite dimensions. Our work on backward stochastic differential equations in infinite dimensions and on other related topics in [1], [2], [3], [4] and [5] uses heavily this result.

In [3] and [4] such a maringale representation theorem is discussed in more generality. Simply, by considering arbitrary right continuous and complete filtration, $\{\mathcal{F}_t\,,\ 0 \leq t \leq T\}$, which is more general than the Wiener filtration which we have here. This new approach is applied in solving some backward stochastic partial differential equations in infinite dimension; cf. [4].

The article is organised as follows. Section 2 contains a preliminary introduction on definition of Wiener processes on Hilbert spaces, stochastic integration and some remarks on what we call the "natural" filtration of a Wiener process. In Section 3 the proofs of the martingale representation theorem(s) are given. Section 4 is devoted to deriving Clark-Ocone theorem.

2. INTRODUCTORY RESULTS ON WIENER PROCESSES AND STOCHASTIC INTEGRATION

Definition 2.1. Let H be a separable Hilbert space. Consider a symmetric positive operator $\mathcal{Q} : H \rightarrow H$, with $tr\ \mathcal{Q} < +\infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that $\{\, W(t) : t \geq 0 \,\}$ is an H-valued stochastic process. We say that $W(\cdot)$ is a $\mathcal{Q}$-Wiener process if it satisfies the following:
(i) $W(0) = 0$ *a.s.*,
(ii) W has continuous sample paths,

(iii) W has independent increments, i.e.

$$\begin{aligned} &\mathbb{P}\,[\,W(t_2)-W(t_1)\in\Gamma_1,\ldots,W(t_{n+1})-W(t_n)\in\Gamma_n\,]\\ &=\prod_{i=1}^{n}\mathbb{P}\,[\,W(t_{i+1})-W(t_i)\in\Gamma_i\,],\end{aligned}$$

for all $0\le t_1<t_2<\ldots<t_{n+1}<\infty$ and $n\ge 1$, where $\Gamma_i\in\mathcal{B}(H)$ for all i, and

(iv) $W(t)-W(s)$ is a Gaussian random variable in H with mean 0 and variance $(t-s)\,\mathcal{Q}$.

Let us consider the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ of subsets of Ω, as $\mathcal{F}_t=\sigma\{W(s),0\le s\le t\}\vee\mathcal{N}$, all $t\ge 0$, where $\mathcal{N}$ is the collection of $\mathbb{P}$-null sets of $\mathcal{F}$. Then condition (iii) is equivalent to the following one:

(iii)$'$ $W(t)-W(s)$ is independent of $\mathcal{F}_s$, for all $0\le s<t<\infty$.

Recall that if W is a $\mathcal{Q}$-Wiener process in H, then $< W(\cdot),h >_H$ is a constant times a 1-dimensional Wiener process, for all $h\in H$. This latter fact is one of the main ingredients needed to define stochastic integration with respect to such Wiener processes.

Let us now present few details on Wiener processes. First note that there exists a complete orthonormal system $\{e_j\}_{j=1}^{\infty}$ in H and a bounded sequence of non-negative real numbers $\{\lambda_j\}_{j=1}^{\infty}$ such that

$$\mathcal{Q}\,e_j=\lambda_j\,e_j\,,\ j=1,2,\ldots \tag{2.1}$$

Thus one can expand $W(t)$ as the following:

$$W(t)=\sum_{j=1}^{\infty}\sqrt{\lambda_j}\;w_j(t)\;e_j, \tag{2.2}$$

where

$$w_j(t)=\frac{1}{\sqrt{\lambda_j}}< W(t)\,,e_j >_H,\ \ j=1,2,\ldots,$$

which are independent real valued Brownian motions. If we define $\tilde{e}_j=\sqrt{\lambda_j}\;e_j,\ \ j=1,2,\ldots,$ then

$$W(t)=\sum_{j=1}^{\infty}w_j(t)\;\tilde{e}_j \tag{2.3}$$

and

$$w_j(t) = \frac{1}{\lambda_j} < W(t), \tilde{e}_j >_H , \; j = 1, 2, \ldots$$

Note that the series (2.3), fortunately, converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ since $\sum_{j=1}^{\infty} \lambda_j = tr\, \mathcal{Q} < \infty$.

On the other hand, note that if $\lambda_j \equiv 1$, for all $j = 1, 2, \ldots$, which is the case when $\mathcal{Q} \equiv \mathrm{I}$, the identity map, then the above series (2.3) becomes $\sum_{j=1}^{\infty} w_j(t)\; e_j$ which will not converge to a genuine process $W(t)$ in H. In this case we call W a (standard) *cylindrical* Wiener process with respect to H; see [8] and [12]. This W can be well defined as a $\mathcal{Q}$-Wiener process in a bigger Hilbert space U such that the inclusion mapping from H to U is Hilbert-Schmidt, cf. [7, Proposition 4.11, p. 96]. However, we will rarely work on this space U.

Let us now define the natural filtration for such cylindrical Wiener processes. We can make use of the formal expansion of W in (2.3) to define the natural filtration for W, to be $\sigma\{w_j(s),\ 0 \leq s \leq t,\ j = 1, 2, \ldots\} \vee \mathcal{N},\ t \geq 0$. We will denote it also by $\mathcal{F}_t$, for $t \geq 0$.

The following remarks on filtrations can be skipped with no harm at a first reading.

Recall that an equivalent definition of a cylindrical Wiener process can also be made when regarding W as a mapping $[0, T] \times H^* \times \Omega \to \mathbb{R},\ (t, l, \omega) \mapsto l \circ W(t, \omega)$ such that $l \circ W(t)$ is a 1- dimensional Wiener process if $|l| = 1$. This enables us to define the following filtration:

$$\mathcal{F}_t(W) = \sigma\{l \circ W(s),\ 0 \leq s \leq t,\ l \in H^*\} \vee \mathcal{N},\ t \geq 0.$$

It can be seen easily after taking limits that $\mathcal{F}_t = \mathcal{F}_t(W)$, for each t.

If we denote by $\mathcal{F}_t^j$ the $\sigma\{w_j(s), 0 \leq s \leq t\} \vee \mathcal{N},\ j = 1, 2, \ldots$, then $\mathcal{F}_t \subseteq \bigvee_{j=1}^{\infty} \mathcal{F}_t^j$. Also, since for each j, $w_j(\cdot) = \int_0^{\cdot} < e_j\, , dW(s) >_H$, then we conclude that $\bigvee_{j=1}^{\infty} \mathcal{F}_t^j \subseteq \mathcal{F}_t$, for each t. In particular, we have $\mathcal{F}_t = \bigvee_{j=1}^{\infty} \mathcal{F}_t^j$, for each t.

On the other hand, note that since W is cylindrical it can be written formally, as in (2.3), as an infinite sum $\sum_{j=1}^{\infty} w_j(\cdot)\, e_j$. Define $W^N = \sum_{j=1}^{N} w_j\, e_j$ and let $\mathcal{F}_t^{(N)}$ be the σ - algebra of subsets of Ω, generated by

$$\{w_j(s) :\ 0 \le s \le t,\ j = 1, 2, \dots N\}.$$

Then one can easily deduce from the definitions that $\mathcal{F}_t^{(N)} = \bigvee_{j=1}^{N} \mathcal{F}_t^j = \sigma\{W^N(s),\ s \le t\} \vee \mathcal{N}$, for all t. Hence $\mathcal{F}_t = \bigvee_{N=1}^{\infty} \mathcal{F}_t^{(N)}$, for all t.

Let us now suppose that we are having an *abstract Wiener space* (*A.W.S.*), $\imath : H \to E$, i.e. H is a separable Hilbert space included in a Banach space E via $\imath$ which is a continuous injective map with dense image and $\gamma -$ *radonifying*, i.e. the push-forward measure $\imath_*(\gamma^H) = \gamma$ is a genuine measure on E, called the Wiener measure on E, where γ^H is the canonical (Gaussian) cylindrical set measure on H. As usual by identifying H with its dual, there is the adjoint of $\imath$, $\jmath \equiv \imath^* : E^* \to H$, such that $\jmath(E^*)$ is dense in H with respect to $L^2(E, \gamma\,; \mathbb{R})$. Here $L^2(E, \gamma\,; \mathbb{R})$ denotes $L^2(E, \mathcal{B}(E), \gamma\,; \mathbb{R})$. Moreover, if $l \in E^*$ and $h \in H$, then $l \circ \imath(h) = \,< h\,, \jmath(l) >_H$.

We should also point out here that if $\imath : H \to E$ is an A.W.S., then $\mathcal{F}_t = \mathcal{F}_t(\tilde{W})$, where $\mathcal{F}_t(\tilde{W}) \equiv \sigma\{\tilde{W}(s), s \le t\} \vee \mathcal{N}$ and $\tilde{W} \equiv \imath(W)$ which is a genuine Wiener process taking values in E. To see this, first note that $\tilde{W}(t)$ is $\mathcal{F}_t$ measurable, for each t, as seen from the definition of $\tilde{W}(t)$ as the sum $\tilde{W}(t) = \sum_{j=1}^{\infty} w_j(t)\ \imath(e_j)$. On the other hand, to see the other inclusion, note that for arbitrary j, $e_j = \lim_{k\to\infty} \jmath(l_k^j)$, for a sequence $\{l_k^j\}_{k\ge 1}$ in E^*. Consider $< \jmath(l_k^j), W >_H : [0,T] \times \Omega \to \mathbb{R}$, $(t,\omega) \mapsto\ < \jmath(l_k^j), W(t,\omega) >_H = l_k^j(\imath(W(t,\omega))) = l_k^j(\tilde{W}(t,\omega))$. This implies that for each k, $< \jmath(l_k^j), W(t) >_H$ is $\mathcal{F}_t(\tilde{W})$ measurable, $\forall\, t$; hence $w_j(t)$ is $\mathcal{F}_t(\tilde{W})$ measurable since $w_j(t) = \int_0^t < e_j\ , dW(s) >_H = \lim_{k\to\infty} l_k^j(\tilde{W}(t))$. Since j is arbitrary the conclusion follows.

Denote by $L_2(H; K)$ the space of Hilbert-Schmidt operators from H into K defined by $L_2(H;K) = \{\Phi \in L(H;K)\ s.t.\ \sum_{j=1}^{\infty} < \Phi\, e_j\ , \Phi\, e_j >_K < \infty\}$. This is a Hilbert space endowed with the norm $|\Phi|_{L_2(H;K)} = (\sum_{j=1}^{\infty} |\Phi\ e_j|_K^2)^{1/2}$

for any arbitrary orthonormal base $\{e_j\}_{j=1}^{\infty}$ of H. For $T < \infty$ and a separable Hilbert space $\tilde{H}$ let $L^2_{\mathcal{F}}(0,T;\tilde{H})$ be the space of all $\{\mathcal{F}_t, 0 \leq t \leq T\}$-progressively measurable processes $\tilde{f}$ with values in $\tilde{H}$, (i.e. for all $t \in [0,T]$, the process $\tilde{f}\,|_{[0,t]\times\Omega}$ is $\mathcal{B}([0,t])\otimes\mathcal{F}_t$-measurable,) such that

$$\mathbb{E}\int_0^T |\tilde{f}(s)|^2_{\tilde{H}}\, ds < \infty.$$

Notice that $L^2_{\mathcal{F}}(0,T;\tilde{H})$ is a Hilbert space with norm

$$|\tilde{f}| = (\mathbb{E}\int_0^T |\tilde{f}(s)|^2_{\tilde{H}}\, ds)^{1/2}.$$

We define a stochastic integral of processes $\Psi \in L^2_{\mathcal{F}}(0,T;L_2(H;K))$ by approximation as follows

$$\int_0^T \Psi(s)dW(s) := \lim_{N\to\infty}\sum_{j=1}^N \int_0^T (\Psi(s)\, e_j)\, dw_j(s), \tag{2.4}$$

where the integral in the right hand side now makes sense as a stochastic integral with respect to 1-dimensional Brownian motions. The limit in (2.4) exists $\mathbb{P}$- $a.s.$ since

$$\mathbb{E}\ \Big|\ \sum_{j=1}^N \int_0^T (\Psi(s)\, e_j)\, dw_j(s)\ \Big|^2_K = \sum_{j=1}^N \mathbb{E}\int_0^T |\ \Psi(s)\, e_j\ |^2_K\, ds$$

$$\to \sum_{j=1}^{\infty} \mathbb{E}\int_0^T |\ \Psi(s)\, e_j\ |^2_K\, ds\ < \infty, \tag{2.5}$$

as $N \to \infty$. Thereby the integral $\int_0^T \Psi(s)\, dW(s)$ is well-defined and belongs to $L^2(\Omega,\mathcal{F}_T,\mathbb{P};K)$. Furthermore, in fact, we have

$$\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{j=1}^N \int_0^t (\Psi(s)\, e_j)\, dw_j(s) - \int_0^t \Psi(s)\, dW(s)\ \Big|^2_K \to 0,$$

as $N \to \infty$. Thus $\int_0^{\cdot} \Psi(s)\ dW(s)$ can also be constructed as a limit in the above respect and is a square integrable martingale with values in K. Other equivalences and extensions of this definitions can be found in the literature, e.g. [11], [12] and [16].

On the other hand, recall that almost the same definition can also be made for the case when W is a $\mathcal{Q}$-Wiener process. See [7], [11], [12] and [16] for clear treatment of this subject and also for the definition of martingales in infinite dimensions.

3. REPRESENTATION OF MARTINGALES

In this section we prove infinite dimensional versions of the well-known finite dimensional martingale representation theorem; cf. e.g. [15]. We will study the two cases when having a cylindrical and a genuine Wiener process in H.

The main theorem of this section is the following.

Theorem 3.1. *Let $\{M(t), 0 \leq t \leq T\}$ be a square integrable martingale in K with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, i.e. $\sup_{0\leq t\leq T} \mathbb{E}\,|M(t)|_K^2 < \infty$. Then there is a unique stochastic process $R \in L^2_{\mathcal{F}}(0,T; L_2(H;K))$ such that, for all $0 \leq t \leq T$, we have a.s.*

$$M(t) = M(0) + \int_0^t R(s)\, dW(s). \tag{3.1}$$

In particular, M has a continuous modification.

Note that $M(0)$ in (3.1) equals to $\mathbb{E}\,(M(t))$, for all t.

Pardoux in [14] has stated this theorem without a proof.

A trivial case of such a martingale M in the theorem is $\mathbb{E}\,[\int_0^T R(s)\, dW(s) |\, \mathcal{F}_t]$, for $0 \leq t \leq T$.

Before introducing the proof let us present some notations which we will need. For $n \in \mathbb{N}$ denote by $\pi_n : H \to H_n$, the orthogonal projection from the space H onto the finite dimensional space $H_n \equiv < e_1, \ldots, e_n > \cong \mathbb{R}^n$, which is generated by the first n elements of the basis $\{e_j\}_{j\geq 1}$ of H.

Lemma 3.1. *If $F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, then there exists a sequence $\{F^N\}_{N=1}^\infty$ of random variables such that every F^N is $\mathcal{F}_T^{(N)}$ - measurable and $\mathbb{E}\,|F - F^N|_{\mathbb{R}}^2 \to 0$, as $N \to \infty$.*

Proof. Take $F^N \equiv \mathbb{E}\,[\, F |\, \mathcal{F}_T^{(N)}]$. Then since $\sup_N \mathbb{E}\,|F^N|^2 < \infty$, $\{F^N, N = 1, 2, \ldots\}$ is a uniformly integrable martingale. Thus F^N converges to F *a.s.*

and in L^1 as $N \to \infty$. Being $\{F^N\}_{N\geq 1}$ bounded in L^2 implies that this convergence holds also in L^2, as this can be seen from [15, Theorem 3.1, p. 86]. $\square$

For example, let for simplicity only K be $\mathbb{R}$ and assume that F is the real valued random variable $\int_0^T R(s)\, dW(s)$, where $R \in L^2_{\mathcal{F}}(0,T;L_2(H;\mathbb{R}))$. It is easy to compute F^N, defined in the above lemma explicitly, and show that F^N attains actually a formula like (3.2) below (with $\mathbb{E}\,[F^N] = 0$ and $R^N \equiv \mathbb{E}\,[\,R|\,\mathcal{F}_T^{(N)}]$).

Proof of Theorem 3.1. Let us first consider the one dimensional case, i.e. when M takes values in $\mathbb{R}$. Let $F \equiv M(T)$. From Lemma 3.1 we can approximate F by $F^N \in L^2(\Omega, \mathcal{F}_T^{(N)}, \mathbb{P}; \mathbb{R})$ such that $\mathbb{E}\,|F - F^N|^2_{\mathbb{R}} \to 0$ as $N \to \infty$.

From [15, Proposition 3.2, p. 199] (or see [10, Lemma 2.4, p. 237]) we see that, for all $N \geq 1$, there exists a unique stochastic process $R^N \in L^2_{\mathcal{F}^{(N)}}(0,T;L_2(H_N;\mathbb{R}))$ such that

$$F^N = \mathbb{E}\,[F^N] + \int_0^T R^N(s)\, dW^N(s). \tag{3.2}$$

By letting $\tilde{R}^N \triangleq R^N \circ \pi_N$, which then belongs to $L^2_{\mathcal{F}}(0,T;L_2(H;\mathbb{R}))$, we can re-write (3.2) as

$$F^N = \mathbb{E}\,[F^N] + \int_0^T \tilde{R}^N(s)\, dW(s). \tag{3.3}$$

We want to obtain such a representation for F. Note that from (3.3) we derive the following

$$\mathbb{E}\int_0^T |\tilde{R}^m(s) - \tilde{R}^N(s)|^2_{L_2(H;\mathbb{R})}\, ds =$$
$$\mathbb{E}\,|F^m - F^N - \mathbb{E}\,F^m + \mathbb{E}\,F^N|^2_{\mathbb{R}} \longrightarrow 0,$$

as $m, N \to \infty$. Therefore $\{\tilde{R}^N\}_{N=1}^{\infty}$ is a Cauchy sequence in $L^2_{\mathcal{F}}(0,T;L_2(H;\mathbb{R}))$, whence it has a limit in this space. Call it R. Finally, by passing the L^2 -

limit through in (3.3) as $N \to \infty$, we get

$$F = \mathbb{E}\,[F] + \int_0^T R(s)\, dW(s),$$

which is the required formula. Hence

$$M(t) = \mathbb{E}\,[\,F \mid \mathcal{F}_t\,] = M(0) + \int_0^t R(s)\, dW(s).$$

It remains to prove such a representation when M takes values in the space K.

Assume for simplicity that $\mathbb{E}\,[M] = 0$. If M is taking values in K and $\{\acute{e}_l\}_{l=1}^{\infty}$ is an arbitrary orthonormal basis of K, then $< M, \acute{e}_l >_K$ is square integrable martingale in $\mathbb{R}$ for each l. Hence

$$\begin{aligned} M(t) &= \sum_{l=1}^{\infty} < M(t), \acute{e}_l >_K \acute{e}_l \\ &= \sum_{l=1}^{\infty} \int_0^t R_{\acute{e}_l}(s)\, dW(s)\, \acute{e}_l\,, \end{aligned}$$

for some unique $R_{\acute{e}_l} \in L^2_{\mathcal{F}}(0,T; L_2(H;\mathbb{R}))$.

Now

$$\begin{aligned} +\infty > \mathbb{E}\,|M(t)|_K^2 &= \mathbb{E}\sum_{l=1}^{\infty} \int_0^t |R_{\acute{e}_l}(s)|^2_{L_2(H;\mathbb{R})}\, ds \\ &= \mathbb{E}\int_0^t |\sum_{l=1}^{\infty} R_{\acute{e}_l}(s)\, \acute{e}_l\, |^2_{L_2(H;K)}\, ds \\ &= \mathbb{E}\int_0^t |R(s)\, |^2_{L_2(H;K)}\, ds, \end{aligned}$$

where $R(s) \triangleq \sum_{j=1}^{\infty} R_{\acute{e}_l}(s)\, \acute{e}_l$. This completes the proof. □

Example 3.1. Assume that F is the random variable defined by

$$F := f\, exp\{ \int_0^T < R(s), dW(s) >_H - \frac{1}{2}\int_0^T |R(s)|^2_H\, ds\},$$

for $R \in L^2_{\mathcal{F}}(0,T;H)$ (e.g. $R = h \in L^2([0,T];H)$), where f is a fixed element of K. By applying Itô's formula (see e.g. [7]) to the process

$$M(t) := f\, exp\{ \int_0^t < R(s), dW(s) >_H - \frac{1}{2} \int_0^t |R(s)|^2_H \, ds\}, \;\; 0 \leq t \leq T,$$

we have $\mathbb{P}$ - a.s.

$$\begin{aligned} M(t) &= f + \int_0^t M(s) < R(s)\, dW(s) >_H \\ &= f + \int_0^t \tilde{R}(s)\, dW(s), \end{aligned}$$

for all t, where $\tilde{R}(\cdot) := M(\cdot)R(\cdot)$ which belongs to $\in L^2_{\mathcal{F}}(0,T;L_2(H;K))$ after identifying R with its corresponding element in $L_2(H;\mathbb{R})$. Therefore $\{M(t)\,,\, 0 \leq t \leq T\}$ is a martingale in K. This gives the representation of M and $F(\equiv M(T))$ in the sense of the preceding theorem.

Note that Theorem 3.1 applies also when having a $\mathcal{Q}$ - Wiener process instead of a cylindrical one. We record this in the following corollary.

Corollary 3.1. *Suppose that W is a $\mathcal{Q}$ - Wiener process evolving in H. Let $\{M(t),\ 0 \leq t \leq T\}$ be a square integrable martingale in K with respect to the natural filtration of W, $\{\mathcal{F}_t\}_{t\geq 0}$. Then there is a unique stochastic process $R \in L^2_{\mathcal{F}}(0,T;L^{\mathcal{Q}}_2(H;K))$ such that, for all $0 \leq t \leq T$, we have $\mathbb{P}$ - a.s.*

$$M(t) = M(0) + \int_0^t R(s)\, dW(s).$$

In particular, M has a continuous modification.

Here $G \in L^{\mathcal{Q}}_2(H;K) \Leftrightarrow G\mathcal{Q}^{1/2} \in L_2(H;K)$.

Proof. Note that W is a cylindrical Wiener process on $\tilde{H} \equiv \mathcal{Q}^{1/2}(H)$, equipped with the inner product

$$< a, b >_{\tilde{H}} := < \mathcal{Q}^{-1/2}a\,,\mathcal{Q}^{-1/2}b >_H,$$

as this can be seen easily from expanding W as an infinite sum as we did earlier in Section 2. By using Theorem 3.1, the result then follows. □

4. CLARK-OCONE THEOREM

We have seen in Section 3 that the martingale representation theorem holds in infinite dimensional setting. We may ask now if a *Clark-Ocone* formula still holds in this setting. The answer is positive and we shall see below how we can find the process R appearing in (3.1) if the terminal value $M(T)$ is regular enough. Before going directly to this business let us present the following notions that we shall need.

Suppose that $\imath : H \to E$ is an A.W.S. with γ being the Wiener measure on E. Let $\mathcal{E} \equiv C_0([0,T];E)$ and $\mathcal{H} \equiv L_0^{2,1}([0,T];H)$. Then $\mathcal{I} : \mathcal{H} \to \mathcal{E}$ is an A.W.S., where $\mathcal{I}(h)(t) = \imath(h(t))$ if $h \in \mathcal{H}$; cf. [6]. Denote by Γ the corresponding Wiener measure on $\mathcal{E}$. Let J denote $I^* : \mathcal{E}^* \to \mathcal{H}$. Assume that $\{W(t), 0 \leq t \leq T\}$ is a cylindrical Wiener process on H. We have seen earlier that $\mathcal{F}_t = \mathcal{F}_t(\tilde{W})$, for all $0 \leq t \leq T$, where $\tilde{W} \equiv \imath(W)$. Now since the Borel σ-algebra of $\mathcal{E}$ is also the σ-algebra generated by all $l \in \mathcal{E}^*$, then $\mathcal{B}(\mathcal{E}) = \mathcal{F}_T(\tilde{W})$. In particular, $\mathcal{B}(\Omega) = \mathcal{F}_T$, where Ω is set to be $\mathcal{E}$ from here on.

Since $\mathcal{E}^*$ is dense in $\mathcal{H}$, if $h \in \mathcal{H}$ then there exists a sequence $\{l_k\}_{k\geq 1}$ in $\mathcal{E}^*$ such that $h = \lim_{k\to\infty} J(l_k)$ in $\mathcal{H}$. But $|J(l_k)|_{\mathcal{H}} = |l_k|_{L^2(\Omega,\Gamma;\mathbb{R})}$, as it is well-known from the construction of abstract Wiener spaces. Thus $\{l_k\}_{k\geq 1}$ converges in $L^2(\Omega,\Gamma;\mathbb{R})$. Let $W(h) := \lim_{k\to\infty} l_k$. Note that if $h = J(l)$, some $l \in E^*$, then $W(J(l)) = l$ *a.s.* But $l(I(\acute{h})) = < j(l), \acute{h} >_{\mathcal{H}}$ for all $\acute{h} \in \mathcal{H}$. Thus we obtain $W(J(l))(I(\acute{h})) = < J(l), \acute{h} >_{\mathcal{H}}$. In particular, $W(h)$ generalises the inner product $< h , \cdot >_{\mathcal{H}}$ on $\mathcal{H}$.

On the other hand, for any $h \in \mathcal{H}$, $W(h) = \int_0^T < \dot{h}(s) , dW(s) >_H$ *a.s.*, the *Paley-Wiener* integral; cf. e.g. [18, p. 266]. Some of these remarks were discussed in [19], see also [13] and [17]. Other notation of $W(h)$ appeared in the literature as δh, $< h, -\tilde{>}_{\mathcal{H}}$ and $I(h)$. For more information see [9].

Definition 4.1. Let Ξ be a real separable Hilbert space. A function $F : \Omega \to \Xi$, is called an Ξ-valued cylindrical polynomial if it is of the form

$$F = p(W(h_1), \ldots, W(h_n)), \quad h_j \in \mathcal{H},$$

where $p(\underline{x}) = \sum_{i=1}^n p^i(\underline{x})\ \xi_i$, where p^i, $i \geq 1$, are real-valued polynomials on $\mathbb{R}^n$, and $\{\xi_i\}_{i\geq 1}$ is an orthonormal base of Ξ. Therefore p is a linear combination of functions $x^m \ \xi$, where $m > 0, x \in \mathbb{R}$ and $\xi \in \Xi$. The totality of such polynomials will be denoted by $\mathcal{P}(\Xi)$.

It is well-known (see [20]) that $\mathcal{P}(K)$ is dense in $L^p(\Omega, \Gamma; K)$ for all $1 \leq p < \infty$.

Define the $\mathcal{H}$ - *gradient* of such $F = p(W(h_1), \ldots, W(h_n))$ by

$$\nabla_{\mathcal{H}} F = \sum_{j=1}^{n} \partial_j p(W(h_1), \ldots, W(h_n)) \otimes \int_0^{\cdot} \dot{h}_j(s) ds.$$

The presence of this tensor $\otimes$ is to regard $\nabla_{\mathcal{H}}$ as a random variable $\Omega \to K \otimes \mathcal{H}$. This is denoted also by ∇F and is called the *Gross-Sobolev derivative* of F; see [18] and [17] for the properties.

Define for $h \in \mathcal{H}$,

$$\nabla_h F = \sum_{j=1}^{n} \partial_j p(W(h_1), \ldots, W(h_n)) < h_j \, , h >_{\mathcal{H}} .$$

Thus, for fixed ω, $\nabla_{\cdot} F(\omega) : \mathcal{H} \to K$ is a continuous linear operator.

Remark 4.1. For $F \in \mathcal{P}(K)$, define $DF(\omega)(u) = \frac{\partial}{\partial \lambda} \ F(\omega + \lambda u)|_{\lambda=0} (\in K)$, $\omega, u \in \Omega$. Then for $h \in \mathcal{H}$,

$$\begin{aligned} D_{\mathcal{H}} F(\omega)(h) \ &\equiv \ DF(\omega)(\imath(h)) \\ &= \ \sum_{j=1}^{n} \partial_j p(W(h_1), \ldots, W(h_n)) < h_j \, , h >_{\mathcal{H}} \ . \end{aligned}$$

In particular,

$$\nabla_h F(\omega) = [\nabla_{\mathcal{H}} F(\omega) \, , h]_{\mathcal{H}} = D_{\mathcal{H}} F(\omega)(h).$$

This "inner product", $[\cdot \, , \cdot]_{\mathcal{H}}$, is a bilinear map from $(K \otimes \mathcal{H}) \times \mathcal{H}$ to K, defined by taking the inner product of the corresponding $\mathcal{H}$ - valued of the first entry with the second entry to obtain an element of K. One could look at it as the following:

$$[k \otimes h' \, , h]_{\mathcal{H}} = k \otimes < h' \, , h >_{\mathcal{H}} = k < h' \, , h >_{\mathcal{H}} \, ,$$

for $h, h', k \in \mathcal{H}$. This $[\cdot \, , \cdot]_{\mathcal{H}}$ agrees with the inner product $< \cdot \, , \cdot >_{\mathcal{H}}$ in the case $K = \mathbb{R}$.

Define

$$D_tF(\omega) = (1_K \otimes \frac{d}{dt})\,\{\nabla_{\mathcal{H}}F(\omega)_t\}.$$

Hence D_tF is a mapping from $\Omega \to K \otimes H$ and

$$D_tF = \sum_{j=1}^{n} \partial_j p(W(h_1), \ldots, W(h_n)) \otimes \dot{h}_j(t).$$

Observe that $\nabla_{\mathcal{H}} : \mathcal{P}(K) \to \mathcal{P}(K \otimes \mathcal{H})$ and similarly $\nabla^k_{\mathcal{H}} : \mathcal{P}(K) \to \mathcal{P}(K \otimes \mathcal{H}^{\otimes k})$ for $k \geq 1$. This operator $\nabla_{\mathcal{H}}$ is closeable on all $L^p(\Omega, \Gamma; \mathbb{R})$ spaces, $1 \leq p < \infty$; see [18, p. 265]. Thus we can define the spaces $\mathbb{D}_{p,k}(K)$ to be the completion of $\mathcal{P}(K)$ under the following norm:

$$||F||_{p,k} = (\mathbb{E}\,|F|^p_K + \mathbb{E}\,|\nabla^k_{\mathcal{H}}F|^p_{K\hat{\otimes}_2\mathcal{H}^{\hat{\otimes} k}})^{1/p},$$

where $\hat{\otimes}_2$ and $\hat{\otimes}$ denote the completed Hilbert-Schmidt tensor product. Thus $\nabla_{\mathcal{H}} : \mathbb{D}_{p,k}(K) \to \mathbb{D}_{p,k-1}(K\hat{\otimes}_2\mathcal{H}^{\hat{\otimes}(k-1)})$ is well-defined as a linear operator. In particular, $F \in \mathbb{D}_{p,1}(K)$ if and only if there exists a sequence of cylindrical (smooth or polynomial) random variables $\{F_n : n \in \mathbb{N}\}$ converging to F in $L^p(\Omega, \Gamma; K)$ such that $\nabla_{\mathcal{H}}F_n$ is Cauchy in $L^p(\Omega, \Gamma; K\hat{\otimes}_2\mathcal{H})$; from which $\nabla_{\mathcal{H}}F = \lim_{n\to\infty}\nabla_{\mathcal{H}}F_n$.

From the definition of D_t we conclude easily that D_t is also well-defined as a linear map that takes every $F \in \mathbb{D}_{p,k}(K)$ to $D_t\,F \in \mathbb{D}_{p,k-1}(K\hat{\otimes}_2H^{\hat{\otimes}(k-1)})$, so $\nabla^k_{\mathcal{H}}$ and $D^k_{t_1,\ldots,t_k} \equiv D_{t_1}D_{t_2}\ldots D_{t_k}$ make sense on their relevant spaces. We shall let here $p = 2$ and $k = 1$. Notice that

$$||F||_{2,1} = (\mathbb{E}\,|F|^2_K + \mathbb{E}\int_0^T |D_tF|^2_{K\hat{\otimes}_2H}\,dt)^{1/2}.$$

The inner product $[\cdot,\cdot]_{\mathcal{H}}$, defined earlier, can easily be extended by linearity to a continuous bilinear map : $(K\hat{\otimes}_2\mathcal{H}) \times \mathcal{H} \to K$. We will denote it also by $[\cdot,\cdot]_{\mathcal{H}}$. Similarly, this definition of $[\cdot\,,\cdot]_{\mathcal{H}}$ also can be made with respect to any arbitrary separable Hilbert space, e.g. the space H.

Let us try to make use of the above definitions in the following example when dealing with the classical Wiener space.

Example 4.1. Let K and H be $\mathbb{R}$. Consider $F = f(W(t_1), \ldots, W(t_n))$, $f \in C^\infty(\mathbb{R}^n)$. For $h \in \mathcal{H}$, we have

$$\mathbb{E} < \nabla_{\mathcal{H}} F , h >_{\mathcal{H}} = \mathbb{E} [\nabla_h F] = \mathbb{E} [F . \int_0^T \dot{h}(s) \, dW(s)],$$

using the Cameron-Martin theorem; cf. [13] or [17, p. 10]. Here we have identified the elements of $\mathcal{H}\hat{\otimes}_2\mathbb{R}$ with the corresponding ones in $\mathcal{H}$.

Note that $W(t_i)$ can be written as $W(h_i)$, where $h_i = \int_0^{\cdot} 1_{[0,t_i]}(s) ds$, $i = 1, 2, \ldots, n$. Direct computations show then that

$$\nabla_{\mathcal{H}} F = \sum_{j=1}^{n} \partial_i f(W(t_1), \ldots, W(t_n)) \cdot \int_0^{\cdot} 1_{[0,t_i]}(s) ds,$$

$$D_t F = \sum_{j=1}^{n} \partial_i f(W(t_1), \ldots, W(t_n)) \cdot 1_{[0,t_i]}(t),$$

$$\nabla_h F = \sum_{j=1}^{n} \partial_i f(W(t_1), \ldots, W(t_n)) \int_0^T < 1_{[0,t_i]}(s), \dot{h}(s) >_{\mathbb{R}} \; ds.$$

Finally, $D_t W(s, \omega) = 1_{[0,s]}(t)$. In particular, $D_t W(s, \omega) = 0$ if $t > s$. Similar calculations can be found in [13] and [17].

We are now ready to state the Clark-Ocone theorem.

Theorem 4.1 (Clark-Ocone Theorem). *If* $F \in \mathbb{D}_{2,1}(K)$, *then*

$$F = \mathbb{E} [F] + \int_0^T [\, \mathbb{E}\{D_t F | \mathcal{F}_t\} \, , dW(t) \,]_H . \tag{4.1}$$

Proof. Since $\mathcal{P}(K)$ is dense in $\mathbb{D}_{2,1}(K)$, it is sufficient to prove the theorem for elements of $\mathcal{P}(K)$. Suppose that $F \in \mathcal{P}(K)$. Let $0 \leq t_j < t_{j+1} \leq T$ and α_j be bounded H - valued and $\mathcal{F}_{t_j}$ −measurable. Then $k(t) := (t \wedge t_{j+1} - t \wedge t_j) \, \alpha_j$ is a bounded, $\mathcal{F}_*$ - adapted process, with paths taking values in $\mathcal{H}$. By Cameron-Martin theorem

$$\mathbb{E} [\, F(\omega + \tau \mathcal{I}(k(\cdot))) \cdot \exp(-\tau \int_0^T \; < \; \dot{k}(s) \, , dW(s) >_H$$

$$- \frac{\tau^2}{2} \int_0^T |\dot{k}(s)|_H^2 ds) \,] \; = \; \mathbb{E} [F]. \tag{4.2}$$

Differentiating (4.2) for τ at $\tau = 0$ yields the following

$$\mathbb{E}\,[\nabla_{\mathcal{H}} F, k]_{\mathcal{H}} = \mathbb{E}\,[F \cdot \int_0^T < \dot{k}(s)\,, dW(s) >_H]. \tag{4.3}$$

By linearity (4.3) holds for any bounded elementary $H-$valued process, k, adapted to $\{\mathcal{F}_t\,,\; 0 \le t \le T\}$.

Assume that $\mathbb{E}\;[F] = 0$ for simplicity, or consider $F - \mathbb{E}\;[F]$. Let g be a bounded elementary process with values in H, adapted to $\{\mathcal{F}_t\,,\; 0 \le t \le T\}$. Take $c \in \mathbb{R}$ and let

$$G = c + \int_0^T < g(s)\,, dW(s) >_H .$$

Thus $G \in L^2(\Omega, \Gamma; \mathbb{R})$. By Theorem 3.1 such G are dense in $L^2(\Omega, \Gamma; \mathbb{R})$. We can observe immediately, as done for (4.3), that

$$\begin{aligned}
\mathbb{E}\,[\,F \cdot G\,] &= \mathbb{E}\,[\,\nabla_{\mathcal{H}} F\,, \int_0^{\cdot} g(s)\,ds\,]_{\mathcal{H}} \\
&= \mathbb{E} \int_0^T [\,D_s F\,, g(s)\,]_H ds \\
&= \mathbb{E} \int_0^T [\,\mathbb{E}\{D_s F | \mathcal{F}_s\}\,, g(s)\,]_H ds \\
&= \mathbb{E}\,(\int_0^T [\,\mathbb{E}\{D_s F | \mathcal{F}_s\}\,, dW(s)\,]_H \cdot \int_0^T [\,g(s)\,, dW(s)\,]_H\,) \\
&= \mathbb{E}\,[\int_0^T [\,\mathbb{E}\{D_s F | \mathcal{F}_s\}\,, dW(s)\,]_H \cdot G\,].
\end{aligned}$$

The proof is complete. □

As a result of this theorem let us offer a direct proof for *Poincaré inequality*; see [18, Lemma B.8.1, p.284] for another proof.

Corollary 4.1. *For $F \in \mathbb{D}_{2,1}(K)$ the following holds*

$$\mathbb{E}\,|\,F - \mathbb{E}\,[F]\,|_K^2 \le \mathbb{E}\,|\nabla_{\mathcal{H}} F|^2_{K \hat{\otimes}_2 \mathcal{H}}.$$

Proof. From Theorem 4.1 we conclude that

$$\begin{aligned}
\mathbb{E} \mid F - \mathbb{E}[F] \mid_K^2 &= \mathbb{E} \mid \int_0^T [\, \mathbb{E}[\, D_t F | \mathcal{F}_t \,] \,, dW(t) \,]_H \mid_K^2 \\
&\leq \mathbb{E} \int_0^T | \mathbb{E}[\, D_t F | \mathcal{F}_t \,] \,|_{K \hat{\otimes}_2 H}^2 \, dt \\
&\leq \mathbb{E} \int_0^T \mid D_t F \mid_{K \hat{\otimes}_2 H}^2 \, dt,
\end{aligned}$$

which completes the proof. □

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