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# ON TESTS OF NEW BETTER THAN RENEWAL USED CLASSES

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ABSTRACT. Based on comparisons of the survival of the renewal variable at age $t \geq 0$ with the survival of its parent variable at $t = 0$, three different renewal classes of life distributions are defined. These renewal classes of life distributions are the new better than renewal used, the new better than renewal used in expectation and the harmonic new better than renewal used in expectation. The corresponding dual classes have been introduced as well. In this paper, we construct test statistics based on the scaled total time on test ($TTT$)- transforms, to test exponentiality versus these renewal classes of life distributions of their duals. The distributions of the test statistics are investigated via simulation for small samples. The power estimates of the test statistics are simulated, for some commonly used distributions in reliability.

## 1. INTRODUCTION

The adverse effects of age on the lifetime (or lifelength) of a device or a system is known by positive ageing or simply ageing and negative ageing simply means non- ageing.

Let $T$ be a non-negative random variable representing the lifetime of a unit with a (continuous) life distribution $F(t)$, where $F(t) = P(T \leq t)$, for which $F(t) = 0$, for $t \leq 0$, density function (assumed to exist) $f(t) = d/dt(F(t))$ and survival (survivorship) function of a new system $\overline{F}(t) = P(T > t) = 1 - F(t)$, for $t \geq 0$.

The corresponding (instantaneous) failure rate of $F$, at time $t$ is defined by

$$\begin{aligned} r_F(t) &= \lim_{x\to 0} \frac{1}{x} P(T \le t+x | T > t), \quad \text{for } t \ge 0. \\ &= \frac{f(t)}{\overline{F}(t)}, \quad \text{for } t \ge 0,\ \overline{F}(t) > 0. \end{aligned}$$

$r_F(t)$ is also known by hazard rate, force of mortality or intensity rate.

Note that, we shall use the terms increasing and decreasing for non-decreasing and non-increasing, respectively.

Suppose that a device (system or component) with lifetime $T$ and a continuous life distribution $F(t)$, is put in operation. The device is replaced upon a failure by a sequence of mutually independent devices. Also, these devices are independent of the first device and identically distributed with the same life distribution $F(t)$. In the long run, the remaining life distribution of the system under operation at time $t$ is given by stationary renewal (or equilibrium) distribution as follows:

$$W_F(t) = \mu_F^{-1} \int_0^t \overline{F}(u)du, \quad \text{for } 0 \le t < \infty, \tag{1.1}$$

where $\mu_F = \int_0^\infty \overline{F}(x)dx < \infty$ is the mean life of the random variable $T$, see Barlow and Proschan (1981), Abouammoh and Ahmed (1992).

The corresponding renewal survival function is given by

$$\begin{aligned} \overline{W}_F(t) &= 1 - W_F(t) \\ &= \mu_F^{-1} \int_t^\infty \overline{F}(u)du, \quad \text{for } 0 \le t < \infty. \end{aligned} \tag{1.2}$$

The density function of the renewal distribution $W_F(t)$, is given by

$$\begin{aligned} w_F(t) &= \frac{d}{dt} W_F(t) \\ &= \mu_F^{-1}\overline{F}(t). \end{aligned} \tag{1.3}$$

The (instantaneous) failure rate of the renewal distribution $W_F(t)$, is given by

$$r_{w_F}(t) = w_F(t)/\overline{W}_F(t). \quad (1.4)$$
$$= \overline{F}(t)/\int_t^{\infty}\overline{F}(u)du,$$

that is

$$r_{w_F}(t) = [\mu_F(t)]^{-1}, \quad (1.5)$$

where $\mu_F(t)$ is the mean remaining life distribution of a used unit at time $t$.

In section 2, we give definitions for three ageing criteria or classes of life distributions. These classes are used in reliability theory, inventory theory, maintenance theory, risk analysis and biometry and have been considered by different authors. In section 3, we present some characterizations of these ageing via the scaled total time on test (TTT)-transform. In section 4, we derive the empirical test statistics for these renewal ageing criteria based on the scaled TTT- transforms. In section 5, small sample studied of these test statistics via simulation are performed. The power estimates of these test statistics are given in section 6, with respect to some commonly used distributions in reliability. Toward the end of this paper, we conclude by some comments and remarks on these results.

## 2. THE RENEWAL CLASSES OF LIFE DISTRIBUTION

Based on comparisons of the survival of the renewal distribution at age $t \geq 0$ with the survival of its parent distribution at $t = 0$ or the survival of the renewal class with its parent (or original) survival at age $t = 0$, we present the definitions of three criteria of positive ageing and their duals.

**Definition 2.1.** a life distribution $F$ with $F(0_-) = 0$, or its survival function $\overline{F}$ is said to have the new better (worse) than renewal used property, denoted by NBRU (NWRU), if

$$\overline{W}_F(x|t) \leq (\geq)\overline{F}(x|0), \quad \text{for } x \geq 0, t \geq 0. \quad (2.1)$$

This is equivalent to

$$P(T_W > t + x | T_W > t) \leq (\geq) P(T > x).$$

This form states that the conditional renewal survival probability of a used system of age $t$ is less (greater) than the survival probability of a new system. Note that

$$\overline{W}_F(x|t) = \overline{W}_F(x+t)/\overline{W}_F(t), \quad \text{for } x > 0, t \geq 0, \overline{W}_F(t) > 0.$$

In fact, relation (2.1) means that the renewal survival at age $x$ for a given time $t$ is less (greater) than the parent survival at age $x$.

This class of life distribution has been introduced by Abouammoh and Qamber (1995). Note that, Cao and Wang (1991), have studied the NBRU(NWRU) property under the name new better (worse) than used in convex ordering, denoted by NBUC(NWUC), see also, Hendi et al. (1993).

**Definition 2.2.** A life distribution $F$ with $F(0_-) = 0$, or its survival function $\overline{F}$ is said to have the new better (worse) than renewal used in expectation property, denoted by NBRUE (NWRUE), if

$$(2.2) \qquad \int_t^\infty \overline{W}_F(x)dx/\overline{W}_F(t) \leq (\geq) \int_0^\infty \overline{F}(x)dx, \quad \text{for } t \geq 0.$$

This relation can have the form

$$E(T_W - t | T_W > t) \leq (\geq) E(T), \quad \text{for } t \geq 0,$$

or

$$(2.3) \qquad \mu_{W_F}(t) \leq (\geq) \mu_F, \quad \text{for } t \geq 0.$$

Note that, the relationship (2.2), can be expressed as

$$(2.4) \quad \int_t^\infty \int_x^\infty \overline{F}(u)dudx \leq (\geq) \mu_F \int_t^\infty \overline{F}(u)du, \quad \text{for } x \geq 0, t \geq 0.$$

The relation (2.3) means that a used system of age $t$ has smaller (bigger) mean remaining renewal life than a new parent system.

**Definition 2.3.** A life distribution $F$ on $(0, \infty)$, with $F(x) = 0$ , for $x \leq 0$, or its survival function $\overline{F}$ is said to have the harmonic new better (worse) than renewal used in expectation property, denoted by HNBRUE (HNWRUE), if

$$(2.5) \qquad \int_t^{\infty} \overline{W}_F(x)dx \leq (\geq)\mu_{W_F} \exp\{-t/\mu_F\}, \quad \text{for } t > 0.$$

Note that, the relationship (2.5) can be expressed as

$$(2.6) \qquad \int_t^{\infty} \int_x^{\infty} \overline{F}(u)dudx \leq (\geq) \exp\{-t/\mu_F\} \int_0^{\infty} \int_x^{\infty} \overline{F}(u)dudx,$$

for $t > 0$.

The last two classes of life distribution were introduced by Abouammoh, Ahmed and Barry (1993).

It is known that the life distribution is said to have new better (worse) than used property, denoted by NBU (NWU), if

$$\overline{F}(x|t) \leq (\geq)\overline{F}(x|0), \quad \text{for } x \geq 0, t \geq 0.$$

One can see that, NBU $\Rightarrow$ NBRU.

Now, we summarize the implications between the renewal classes of life distributions as follows

$$\text{NBRU} \ \longrightarrow \ \text{NBRUE} \ \longrightarrow \ \text{HNBRUE}$$

Similar implications hold for the corresponding dual renewal classes of life distributions. For implications, definitions, properties and the relationship of these renewal classes, see Abouammoh and Qamber (1996) and Abouammoh et al. (1993).

## 3. THE TOTAL TIME ON TEST CONCEPT

In this section, we present some characterizations of ageing via the scaled total time on test (TTT)-transform. The TTT-transform is denoted by $H_F^{-1}(t)$ and the scaled TTT-transform by $\phi_F(t)$. These transforms have been introduced by Barlow and Campo (1975) and Barlow (1979).

Next, we give the following definitions:

**Definition 3.1.** Let $F$ be a (continuous) life distribution (i.e., a distribution function for which $F(t) = 0$, for $t \leq 0$), with survival function $\overline{F}$ and finite mean $\mu_F$. Then

(i) The TTT-transform $H_F^{-1}(t)$ of $F$ is defined by

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \overline{F}(u)du, \quad \text{for } 0 \leq t \leq 1, \tag{3.1}$$

where $F^{-1}(t) = \inf\{x : F(x) \geq t\}$ is the inverse function of $F$.

(ii) The scaled TTT-transform $\phi_F(t)$ of $F$ is defined by

$$\phi_F(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)}, \quad \text{for } 0 \leq t \leq 1 , \tag{3.2}$$

where by equation (3.1), we have $H_F^{-1}(1) = \mu_F$.

**Remark 3.2.** If $F(t) = 1 - \exp(-\lambda t)$, for $t \geq 0$, $\lambda > 0$ then the scaled TTT-transform is given by

$$\phi_F(t) = t, \quad \text{for } 0 \leq t \leq 1. \tag{3.3}$$

Now, we derive the empirical TTT-transform.

Assume $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$ to be an ordered sample from a (continuous) life distribution $F$ (i.e., a distribution function for which $F(t) = 0$, for $t \leq 0$) and $t_{(0)} = 0$.

Further, let

$$z_j = (n - j + 1)(t_{(j)} - t_{(j-1)}), \quad \text{for } j = 1, 2, \ldots, n, \tag{3.4}$$

then

$$S_j = \sum_{k=1}^{j} z_k, \quad \text{for } j = 1, 2, \ldots, n, \tag{3.5}$$

denotes the TTT-transform at $t_{(j)}$, where $S_0 = 0$. Hence

$$w_j = \frac{S_j}{S_n}, \qquad \text{for } j = 1, 2, \ldots, n, \tag{3.6}$$

is an estimate of the scaled TTT-transform, where $S_n = \sum_{j=1}^{n} z_j$, and $w_0 = 0$.

An estimate of the scaled TTT-transform is known as the empirical scaled TTT- transform and obtained as

$$\phi_F\left(\frac{j-1}{n}\right) = w_{j-1}, \qquad \text{for } j = 1, 2, \ldots, n\ . \tag{3.7}$$

The TTT-plot is obtained by plotting $w_j$ against $(j-1)/n$, for $j = 1, 2, \ldots, n$, i.e.,

$$\left(w_{j-1}, \frac{j-1}{n}\right), \qquad \text{for } j = 1, 2, \ldots, n, \tag{3.8}$$

and then connecting the plotted points by straight lines.

## 4. TEST STATISTICS BASED ON THE EMPIRICAL SCALED TTT-TRANSFORMS

In this section, we consider the main problem in testing reliability data, that is to test the null hypothesis

$$H_0 : F(t) = 1 - \exp(-\lambda t), \quad \text{for } t \geq 0, \lambda > 0,$$

versus the alternative

$$H_1 : F(t) = \nabla, \quad \text{(but not exponential)},$$

where $\nabla$ denote the NBRU (NWRU), NBRUE (NWRUE) or HNBRUE (HNWRUE) class of life distributions with non-constant failure rate (positive or negative ageing criteria).

This problem has been considered by many authors for different reliability classes of life distributions. Proschan and Pyke (1967), Barlow (1968), Bickel and Doksum (1969) and Klefsjo (1982) considered this problem for $\nabla$ = IFR. Barlow and Campo (1975) and Bergman (1977) for $\nabla$ = IFRA. Hollander and Proschan (1972), Koul (1977), Deshpande and Kochar (1983) and Kumazawa (1983) for $\nabla$ = NBU. Hollander and Proschan (1975), Koul (1978), Koul and Susarla (1980) and Konjo (1993) for $\nabla$ = NBUE. Klefsjo (1983), Basu and Ebrahimi (1985) for $\nabla$ = HNBUE. Abouammoh and Newby (1989) have considered the problem for $\nabla$ = NBUFR and $\nabla$ = NBAFR.

In the following theorem, we characterize the NBRU (NWRU), NBRUE (NWRUE) and HNBRUE (HNWRUE) properties in terms of the scaled TTT-transforms. This characterization can be utilized to construct two-tailed test statistics for testing $H_0$ (exponentially) against the alternative $H_1(\nabla)$.

**Theorem 4.1.** *Let $F$ be a life distribution with finite mean $\mu_F = \int_0^\infty \overline{F}(u)du$ and let $\phi_F(t)$, for $0 \le t \le 1$ be the corresponding scaled TTT-transform. Then:*

*(i) $F$ is NBRU (NWRU), iff*

$$\phi_F(s) + t(1-\phi_F(s)) \le (\ge)\mu_F^{-1}\int_0^{F^{-1}(s)+F^{-1}(t)} \overline{F}(w)dw, \quad \text{for } 0 \le s \le 1,\ 0 \le t \le 1.$$

*(ii) $F$ is NBRUE (NWRUE), iff*

$$\mu_F \int_{F(t)}^1 [s-\phi_F(s)] \quad \phi'_F \quad (s)(1-s)^{-1}ds \le (\ge)0, \quad \text{for } 0 \le s \le 1,\ 0 \le t \le 1.$$

*(iii) $F$ is HNBRUE (HNWRUE), iff*

$$\int_{F(t)}^1 [1-\phi_F(s)]\phi'_F(s)(1-s)^{-1}ds \le (\ge) \int_0^1 [1-\phi_F(s)]\phi'_F(s)(1-s)^{-1}ds$$
$$\times \exp\{-F(t)/\mu_F\}. \quad \text{for } 0 \le s \le 1,\ 0 \le t \le 1.$$

*Proof.* We prove the theorem for the basic classes, while the proofs for the dual classes are carried out in parallel steps.

(i) The life distribution $F$ has NBRU property, if

$$\overline{W}_F(x|y) \leq \overline{F}(x|0), \quad \text{for } x \geq 0, y \geq 0,$$

or

$$\overline{W}_F(x+y) \leq \overline{F}(x)\overline{W}_F(y), \quad \text{for } x \geq 0, y \geq 0.$$

This is equivalent to

$$\mu_F^{-1} \int_{x+y}^{\infty} \overline{F}(u)du \leq \overline{F}(x)\mu_F^{-1} \int_y^{\infty} \overline{F}(z)dz, \quad \text{for } x \geq 0, y \geq 0.$$

Let $u = y + z$. This gives

$$\int_x^{\infty} \overline{F}(y+z)dz \leq (1 - F(x)) \int_y^{\infty} \overline{F}(z)dz, \quad \text{for } x \geq 0, y \geq 0.$$

The later form leads to

$$[\mu_F - \int_0^{y+x} \overline{F}(u)du] \leq (1 - F(x))[\mu_F - \int_0^y \overline{F}(z)dz], \quad \text{for } x \geq 0, y \geq 0.$$

This is equivalent to

$$\mu_F - \int_0^{y+x} \overline{F}(u)du \leq (1 - F(x))[\mu_F - \int_0^y \overline{F}(z)dz],$$

for $x \geq 0, y \geq 0$.

Using the transformation $x = F^{-1}(t)$ and $y = F^{-1}(s)$ and with some simplification, we get

$$1 - \mu_F^{-1} \int_0^{F^{-1}(s)+F^{-1}(t)} \overline{F}(u)du \leq [1 - [\phi_F(s) + t(1 - \phi_F(s)],$$

$$\text{for } 0 \leq s \leq 1. \ 0 \leq t \leq 1.$$

Hence

$$\phi_F(s) + t(1 - \phi_F(s)) \ \leq \ \mu_F^{-1} \int_0^{F^{-1}(s)+F^{-1}(t)} \overline{F}(u)du,$$

$$\text{for } 0 \leq s \leq 1, \ 0 \leq t \leq 1.$$

(ii) The life distribution $F$ has NBRUE property, if

$$\int_t^\infty \overline{W}_F(x)dx \leq \mu_F \overline{W}_F(t), \quad \text{for } x \geq 0, t \geq 0,$$

or

$$\int_t^\infty [\mu_F^{-1} \int_x^\infty \overline{F}(u)du]dx \leq \mu_F [\mu_F^{-1} \int_t^\infty \overline{F}(x)dx], \quad \text{for } x \geq 0, t \geq 0.$$

This is equivalent to

$$\int_t^\infty [\mu_F^{-1} \int_x^\infty \overline{F}(u)du - \overline{F}(x)]dx \leq 0, \quad \text{for } x \geq 0, t \geq 0.$$

Simplifying and using the transformation $x = F^{-1}(s)$ or $F(x) = s$ (i.e., $f(x)dx = ds$ or $dx = ds/[f(F^{-1}(s))]$), yields

$$\int_{F(t)}^1 [s - \mu_F^{-1} \int_0^{F^{-1}(s)} \overline{F}(u)du][f(F^{-1}(s))]^{-1}ds \leq 0,$$

for $0 \leq s \leq 1, 0 \leq t \leq 1$.

This is equivalent to

$$\int_{F(t)}^1 [s - \phi_F(s)][f(F^{-1}(s))]^{-1}ds \leq 0, \quad \text{for } 0 \leq s \leq 1, 0 \leq t \leq 1.$$

Since $[f(F^{-1}(s))]^{-1} = \mu_F \dfrac{\phi'_F(s)}{1-s}$, we have

$$\mu_F \int_{F(t)}^1 [s - \phi_F(s)]\phi'_F(s)(1-s)^{-1}ds \leq 0, \quad \text{for } 0 \leq s \leq 1, 0 \leq t \leq 1.$$

(iii) The life distribution $F$ has HNBRUE property, if

$$\int_t^\infty \overline{W}_F(x)dx \leq \int_0^\infty \overline{W}_F(x)dx \exp\{-t/\mu_F\}, \quad \text{for } x \geq 0, t \geq 0.$$

This is equivalent to

$$\int_t^\infty [\mu_F^{-1} \int_x^\infty \overline{F}(u)du]dx \le \int_0^\infty [\mu_F^{-1} \int_x^\infty \overline{F}(u)du]dx \exp\{-t/\mu_F\},$$

for $x \ge 0, t \ge 0$.

Rearranging the terms and using the transformation $x = F^{-1}(s)$ (i.e., $f(x)dx = ds$ or $dx = ds/[f(F^{-1}(s))]$), yields

$$\begin{aligned} \mu_F \int_{F(t)}^1 [1 &- \mu_F^{-1} \int_0^{F^{-1}(s)} \overline{F}(u)du][f(F^{-1}(s))]^{-1} ds \\ &\le \mu_F \int_0^1 [1 - \mu_F^{-1} \int_0^{F^{-1}(s)} \overline{F}(u)du][f(F^{-1}(s))]^{-1} ds \\ &\quad \times \exp\{-F(t)/\mu_F\}, \quad \text{for } 0 \le s \le 1, 0 \le t \le 1. \end{aligned}$$

Since $[f(F^{-1}(s))]^{-1} = \mu_F \dfrac{\phi_F'(s)}{1-s}$, one has

$$\begin{aligned} \int_{F(t)}^1 [1 - \phi_F(s)]\phi_F'(s)(1-s)^{-1} ds \;&\le\; \int_0^1 [1 - \phi_F(s)]\phi_F'(s)(1-s)^{-1} ds \\ &\quad \times \exp\{-F(t)/\mu_F\}, \end{aligned}$$

for $0 \le s \le 1, 0 \le t \le 1$.

This completes the proof.

**Corollary 4.2.** *If the underlying distribution is the negative exponential with survival $\overline{F}(x) = \exp(-\lambda x)$, $\lambda > 0$, $x \ge 0$, then with inequalities in Theorem 4.1 are replaced by equality signs.*

*Proof.* Relation (i) of Theorem 4.1 with equality sign is given as

$$\phi(s) - t(1 - \phi(s)) = \mu_F^{-1} \int_0^{F^{-1}(s)+F^{-1}(t)} \overline{F}(u)du, \quad 0 \le s \le 1, \; 0 \le t \le 1. \tag{4.1}$$

Since $\overline{F}(x) = e^{-\lambda x}$, $\lambda > 0$, $x \ge 0$, then $F^{-1}(t) = -\lambda^{-1}\log(1-t)$ and $\phi(t) = t$. Hence the R.H.S. of equation (4.1) is given by $s - t(1-s)$. The L.H.S. of equation (4.1) means that

$$\mu_F^{-1} \int_0^{-\lambda^{-1}\log(1-s)(1-t)} e^{-\lambda u} du = s - t(1-s)$$

This proves the first part of the Corollary. The other two parts can be proved in similar steps.

Now, since the TTT-plot $w_{j-1}$ converges to the scaled TTT-transform $\phi_F(t)$, for $0 \leq t \leq 1$ as $n \to \infty$ and $(j-1)/n$ converges to $t$, then the TTT-plot based on an ordered sample $0 = t_{(0)} \leq t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$ behaves as $\phi_F(t)$ does. This suggests the following test statistics based on the scaled TTT-transforms.

### 4.1. THE NBRU TEST STATISTICS

The distribution $F$ is NBRU, iff

$$\phi_F(s) + t(1 - \phi_F(s)) \leq \mu_F^{-1} \int_0^{F^{-1}(s)+F^{-1}(t)} \overline{F}(w)dw, \quad \text{for } 0 \leq s \leq 1,\ 0 \leq t \leq 1.$$

This is equivalent to $U_1(s,t) \leq 0$, where

$$U_1(s,t) = \phi_F(s) + t(1 - \phi_F(s)) - \mu_F^{-1} \int_0^{F^{-1}(s)+F^{-1}(t)} \overline{F}(w)dw, \quad \text{for } 0 \leq s \leq 1, 0 \leq t \leq 1.$$

Integrating both sides of the above relation with respect to $s$ over [0,1] and $t$ over [0,1], one gets

$$W_1 = \int_0^1 \int_0^1 U_1(s,t)dsdt.$$

By using equation (3.7) of section 3, the left hand side of this inequality is estimated, at a specified times $s$ and $t$, by

$$\begin{aligned} D_1(i,j) &= w_{j-1} + n^{-1}(i-1)(1 - w_{j-1}) \\ &\quad -[n\bar{t}]^{-1} \sum_{k=1}^{l_k} (n-k+1)(t_{(k)} - t_{(k-1)}), \end{aligned}$$

where $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n,$ $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ are the ordered statistics of the independent random sample $x_1, x_2, \ldots, x_n$, $t_{(0)} = 0$, $\bar{t}$ is the sample mean of $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ or $x_1, x_2, \ldots, x_n$, and

$$l_k = \begin{cases} [\#t's \leq (t_{(i)} + t_{(j)}) & \text{if } t_{(i)} + t_{(j)} < t_{(n)} \\ n & \text{if } t_{(i)} + t_{(j)} \geq t_{(n)} \end{cases}.$$

Taking the summation of $D_1(i,j)$ over $i$ and $j$, gives the corresponding test statistics $K(n)$, or simply $K$ as

$$\begin{aligned} K &= \sum_{i=1}^{n}\sum_{j=1}^{n}[w_{j-1} + n^{-1}(i-1)(1-w_{j-1})] \\ &\quad -[n\bar{t}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{l_k}(n-k+1)(t_{(k)} - t_{(k-1)}). \end{aligned}$$

To reduce the size of the test statistics, we consider the version

$$K^* = K/n. \tag{4.2}$$

## 4.2. THE NBRUE TEST STATISTICS

The distribution $F$ is NBRUE, iff

$$\mu_F \int_{F(t)}^{1} [s - \phi_F(s)]\phi'_F(s)(1-s)^{-1}ds \leq 0, \quad \text{for } 0 \leq s \leq 1,\ 0 \leq t \leq 1.$$

This is equivalent to

$$\begin{aligned} U_2(F(t)) &= \mu_F \int_{F(t)}^{1} [s - \phi_F(s)]\phi'_F(s)(1-s)^{-1}ds \leq 0 \\ &\quad \text{for } 0 \leq s \leq 1,\ 0 \leq t \leq 1. \end{aligned}$$

Integration of both sides over $[0,1]$ with respect to $z$ (i.e., $z = F(t)$), yields

$$W_2 = \int_0^1 U(z)dz \leq 0,$$

i.e.

$$\begin{aligned} W_2 &= \int_0^1 \left\{\mu_F \int_z^1 [s - \phi_F(s)]\phi'_F(s)(1-s)^{-1}ds\right\} dz \leq 0 \\ &\quad \text{for } 0 \leq s \leq 1,\ 0 \leq z \leq 1. \end{aligned}$$

By using equation (3.7) of section 3, the left hand side of this inequality is estimated at a specified time $t$, by

$$\begin{aligned} D_2(i,j) &= \sum_{i=1}^{n}\sum_{j=i}^{n} n\bar{t}[n^{-1}(j-1)-w_{j-1}](n-j+1)^{-1} \\ &\quad \times(w_{j-1}-w_{j-2})(t_{(j)}-t_{(j-1)})(t_{(i)}-t_{(i-1)}), \end{aligned}$$

where $i = 1,2,\ldots,n$, $j = 1,2,\ldots,n$, $t_{(1)},t_{(2)},\ldots,t_{(n)}$ are the ordered statistics of the independent random sample $x_1,x_2,\ldots,x_n$, $t_{(0)} = 0$ and $\bar{t}$ is the sample mean of $t_{(1)},t_{(2)},\ldots,t_{(n)}$ or $x_1,x_2,\ldots,x_n$.

By further simplification, the equation of $D_2(i,j)$ can give the corresponding test statistics $L(n)$, or simply $L$ as

$$\begin{aligned} L &= \bar{t}\sum_{i=1}^{n}\sum_{j=i}^{n}[(j-1)-nw_{j-1}](n-j+1)^{-1} \\ &\quad \times(w_{j-1}-w_{j-2})(t_{(j)}-t_{(j-1)})(t_{(i)}-t_{(i-1)}), \end{aligned}$$

To reduce the size of the test statistics, we consider the version

$$L^* = L/n. \tag{4.3}$$

## 4.3. THE HNBRUE TEST STATISTICS

The distribution $F$ is HNBRUE, iff

$$\begin{aligned} &\int_{F(t)}^{1}[1 - \phi_F(s)]\phi_F'(s)(1-s)^{-1}ds \\ &\qquad \leq \int_0^1[1-\phi_F(s)]\phi_F'(s)(1-s)^{-1}ds\exp\{-F(t)/\mu_F\}, \\ &\qquad \text{for } 0 \leq s \leq 1,\ 0 \leq t \leq 1. \end{aligned}$$

This is equivalent to

$$\begin{aligned} U_3(F(t)) &= \int_{F(t)}^{1}[1-\phi_F(s)]\phi_F'(s)(1-s)^{-1}ds \\ &\quad -\int_0^1[1-\phi_F(s)]\phi_F'(s)(1-s)^{-1}ds\exp\{-F(t)/\mu_F\} \leq 0, \end{aligned}$$

for $0 \leq s \leq 1, 0 \leq t \leq 1$. Integration of both sides over [0,1] with respect to $z$(i.e., $z = F(t)$), yields

$$W_3 = \int_0^1 U(z)dz \leq 0,$$

i.e.

$$\begin{aligned} W_3 &= \int_0^1 \Big\{\int_z^1 [1-\phi_F(s)]\phi'_F(s)(1-s)^{-1}ds \\ &\quad - \int_0^1 [1-\phi_F(s)]\phi'_F(s)(1-s)^{-1}ds \exp\{-z/\mu_F\}\Big\} dz \leq 0, \\ &\quad \text{for } 0 \leq s \leq 1,\ 0 \leq z \leq 1. \end{aligned}$$

By using equation (3.7) of section 3, the left hand side of this inequality is estimated at a specified time $t$, by

$$\begin{aligned} D_3(i,j) &= n\sum_{i=1}^{n}\{\sum_{j=i}^{n}(1-w_{j-1})(n-j+1)^{-1}(w_{j-1}-w_{j-2})(t_{(j)}-t_{(j-1)}) \\ &\quad -\sum_{j=1}^{n}(1-w_{j-1})(n-j+1)^{-1}(w_{j-1}-w_{j-2})(t_{(j)}-t_{(j-1)}) \\ &\quad \times \exp\{-i/\bar{t}\}\}(t_{(i)}-t_{(i-1)}), \end{aligned}$$

where $i = 1,2,\ldots,n$, $j = 1,2,\ldots,n$, $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ are the ordered statistics of the independent random sample $x_1, x_2, \ldots, x_n$, $t_{(0)} = 0$ and $\bar{t}$ is the sample mean of $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ or $x_1, x_2, \ldots, x_n$.

By further simplification, the equation of $D_3(i,j)$ can give the corresponding test statistics $M(n)$, or simply $M$ as

$$\begin{aligned} M &= n\sum_{i=1}^{n}\{\sum_{j=1}^{n}(1-w_{j-1})(n-j+1)^{-1}(w_{j-1}-w_{j-2})(t_{(j)}-t_{(j-1)}) \\ &\quad -\sum_{j=1}^{n}(1-w_{j-1})(n-j+1)^{-1}(w_{j-1}-w_{j-2})(t_{(j)}-t_{(j-1)}) \\ &\quad \times \exp(-i/\bar{t}\}\}(t_{(i)}-t_{(i-1)}). \end{aligned}$$

To make the test statistic scale invariant or to reduce its size, we consider the version

$$M^* = M/\bar{t}. \tag{4.4}$$

Note that the statistics $K, L$ or $M = 0$, under the null hypothesis $H_0$, see Corollary 4.2, and $K, L$ or $M < (>)0$, under the alternative hypothesis $H_1$ i.e. the underlying distribution if $F$ is NBRU (NWRU), NBRUE (NWRUE) or HNBRUE (HNWRUE), respectively.

## 5. SIMULATION OF SMALL SAMPLES

It is difficult to calculate the exact distributions of the test statistics given by relations (4.1), (4.2) and (4.3). Therefore we would study the performance of these test statistics for small samples which are more common and sufficient for many practical situations. We have simulated the lower and upper percentile points for $\alpha = 0.01, 0.05$ and $0.10$. Tables (5.1), (5.2) and (5.3) give these percentile points of the statistics $K^*, L^*$ and $M^*$ and the calculations are based on 1,000 simulated samples of sizes $n = 2(1)\,10(5)\,30(10)\,50$.

## 6. THE POWER ESTIMATES

For the significance level $\alpha = 0.05$ in Tables (5.1), (5.2) and (5.3), we consider the power of the test statistics $K^*, L^*$ and $M^*$ for some commonly used distributions in reliability modelling. These distributions are linear failure rate, Pareto, Weibull and gamma. Their respective survival functions are the following:

$$\begin{aligned}
\overline{F}(t) &= \exp\left\{-\left(t + \frac{1}{2}\theta t^2\right)\right\}, \quad \text{for } \theta \geq 0, t \geq 0 \\
\overline{F}(t) &= (1 + \theta t)^{-1/\theta}, \quad \text{for } \theta > 0, t \geq 0 \\
\overline{F}(t) &= \exp\{-(t)^\theta\}, \quad \text{for } \theta \geq 0, t \geq 0 \\
\overline{F}(t) &= [\Gamma(\theta)]^{-1} \int_t^\infty x^{\theta-1} \exp(-x)dx, \quad \text{for } \theta \geq 0, t \geq 0
\end{aligned}$$

Note that, the distributions $F_1$ (for $\theta \geq 0$), $F_3$ and $F_4$ (for $\theta \geq 1$) have the NBRU, NBRUE and HNBRUE properties whereas $F_2$ (for $\theta \geq 0$, $F_3$ and $F_4$ (for $\theta \leq 1$) have the NWRU, NWRUE and HNWRUE properties.

In Tables (6.1), (6.2) and (6.3), we give the power estimates of the

$K^*, L^*$ and $M^*$ statistics, respectively. These values of the power estimates are calculated for NBRU (NWRU), NBRUE (NWRUE) and HNBRUE (HNWRUE) distributions obtained by giving some suitable values to the parameter $\theta$. The calculations are based on 1,000 simulated samples of sizes $n = 10, 20, 30$ and significance level $\alpha = 0.05$. It is noted that, the distribution $F_1$ with $\theta = 0$ and the distributions $F_3$ and $F_4$ with $\theta = 1$ coincide with the (negative) exponential distribution $F(t) = \exp(-t)$, for $t \geq 0$. These distributions become more NBRU, NBRUE or HNBRUE as $\theta$ increases. Similarly, the distribution $F_2$ become more NWRU, NWRUE or HNWRUE distribution as $\theta$ decreases.

We note from Tables (6.1), (6.2) and (6.3), that the power estimates of NBRU (NWRU), NBRUE( NWRUE) and HNBRUE (HNWRUE) statistics increase (decrease) as either $\theta$ or the sample size $n$ increases (decreases) for the positive (negative) ageing properties distribution. This means the power estimates increase with the departure from exponentiality.

Also, we note that the departure from exponentiality towards NBRU (NWRU), NBRUE (NWRUE) and HNBRUE (HNWRUE) properties increases as $\theta$ increases.

## 7. CONCLUSION

For NBRU (NWRU), NBRUE(NWRUE) and HNBRUE (HNWRUE) classes of life distributions, the test statistics based on the scaled (TTT)-transforms are developed for testing exponentiality as null hypothesis versus the NBRU (NWRU), NBRUE( NWRUE) and HNBRUE (HNWRUE) ageing properties as the alternative hypothesis. It is noted that, the distribution of $K^*$ is close to symmetry around zero. The distributions of $L^*$ and $M^*$ are relatively skewed towards the right and the left, respectively. For small samples the values of $L^*$ and $M^*$ do not rapidly diverge as the sample size $n$ increases. It is hoped that some suitable transformations for $K^*, L^*$ and $M^*$ statistics can be applied to attain the asymptotic normality of the transformed statistics. The power estimates of the test statistics, as expected, decrease as the NBRU (NWRU), NBRUE( NWRUE) and HNBRUE (HNWRUE) distribution approaches the exponential distribution. The power estimates also decrease as we expected, for example, move from smaller NBRU (NWRU) class to a larger HNBRUE (HNWRUE).

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Table (5.1). Critical Values for the $K^*$ - Statistics.

| n \ α | Lower Percentiles | | | Upper Percentiles | | |
|---|---|---|---|---|---|---|
| | 0.01 | 0.05 | 0.10 | 0.10 | 0.05 | 0.01 |
| 2 | -0.740 | -0.702 | -0.651 | 0.359 | 0.428 | 0.483 |
| 3 | -1.670 | -1.244 | -0.934 | 0.848 | 0.939 | 0.987 |
| 4 | -2.018 | -1.308 | -1.057 | 0.996 | 1.123 | 1.327 |
| 5 | -2.158 | -1.600 | -1.089 | 1.068 | 1.239 | 1.538 |
| 6 | -2.228 | -1.420 | -1.008 | 1.140 | 1.381 | 1.753 |
| 7 | -2.622 | -1.562 | -1.004 | 1.269 | 1.488 | 1.834 |
| 8 | -2.756 | -1.455 | -1.008 | 1.379 | 1.560 | 1.921 |
| 9 | -2.692 | -1.700 | -1.115 | 1.499 | 1.827 | 2.358 |
| 10 | -2.644 | -1.508 | -1.005 | 1.517 | 1.839 | 2.257 |
| 15 | -2.695 | -1.683 | -1.211 | 1.744 | 2.114 | 3.064 |
| 20 | -2.979 | -1.918 | -1.257 | 2.213 | 2.614 | 3.338 |
| 25 | -3.250 | -1.998 | -1.331 | 2.188 | 2.731 | 3.632 |
| 30 | -3.242 | -2.091 | -1.596 | 2.475 | 3.006 | 3.884 |
| 40 | -3.939 | -2.493 | -1.874 | 2.756 | 3.417 | 4.606 |
| 50 | -4.487 | -2.895 | -2.144 | 2.964 | 3.968 | 5.552 |

Table (5.2). Critical Values for the $L^*$ - Statistics.

| n \ α | Lower Percentiles | | | Upper Percentiles | | |
|---|---|---|---|---|---|---|
| | 0.01 | 0.05 | 0.10 | 0.10 | 0.05 | 0.01 |
| 2 | -0.408 | -0.203 | -0.104 | 0.350 | 0.663 | 1.604 |
| 3 | -0.355 | -0.147 | -0.096 | 0.593 | 1.481 | 4.881 |
| 4 | -0.168 | -0.088 | -0.050 | 0.650 | 1.306 | 4.561 |
| 5 | -0.139 | -0.072 | -0.045 | 0.553 | 1.255 | 3.765 |
| 6 | -0.092 | -0.051 | -0.036 | 0.556 | 1.333 | 3.919 |
| 7 | -0.096 | -0.046 | -0.031 | 0.469 | 1.235 | 4.619 |
| 8 | -0.068 | -0.038 | -0.028 | 0.653 | 1.623 | 4.885 |
| 9 | -0.049 | -0.031 | -0.021 | 0.516 | 1.312 | 4.184 |
| 10 | -0.044 | -0.028 | -0.020 | 0.394 | 1.058 | 4.337 |
| 15 | -0.034 | -0.021 | -0.014 | 0.407 | 1.068 | 3.102 |
| 20 | -0.025 | -0.015 | -0.011 | 0.314 | 0.717 | 2.933 |
| 25 | -0.020 | -0.012 | -0.009 | 0.249 | 0.713 | 3.010 |
| 30 | -0.015 | -0.010 | -0.008 | 0.192 | 0.410 | 2.242 |
| 40 | -0.013 | -0.008 | -0.006 | 0.157 | 0.366 | 1.679 |
| 50 | -0.009 | -0.007 | -0.005 | 0.139 | 0.319 | 1.431 |

Table (5.3). Critical Values for the $M^*$ - Statistics.

| n \ ∝ | Lower Percentiles 0.01 | 0.05 | 0.10 | Upper Percentiles 0.10 | 0.05 | 0.01 |
|---|---|---|---|---|---|---|
| 2 | -6.007 | -2.584 | -1.595 | 0.042 | 0.058 | 0.070 |
| 3 | -6.194 | -2.638 | -1.687 | 0.221 | 0.395 | 0.992 |
| 4 | -4.637 | -2.466 | -1.566 | 0.237 | 0.455 | 1.208 |
| 5 | -6.819 | -2.800 | -1.860 | 0.233 | 0.390 | 0.939 |
| 6 | -5.342 | -2.834 | -1.574 | 0.192 | 0.375 | 0.884 |
| 7 | -4.790 | -2.301 | -1.525 | 0.216 | 0.362 | 1.025 |
| 8 | -5.876 | -2.539 | -1.677 | 0.189 | 0.352 | 0.865 |
| 9 | -3.991 | -2.261 | -1.380 | 0.172 | 0.352 | 0.865 |
| 10 | -4.043 | -2.069 | -1.477 | 0.118 | 0.280 | 0.874 |
| 15 | -3.715 | -2.168 | -1.527 | 0.082 | 0.210 | 0.502 |
| 20 | -3.819 | -1.827 | -1.387 | 0.068 | 0.156 | 0.593 |
| 25 | -3.036 | -1.808 | -1.328 | 0.057 | 0.129 | 0.430 |
| 30 | -3.066 | -1.802 | -1.262 | 0.036 | 0.078 | 0.221 |
| 40 | -3.517 | -1.954 | -1.427 | 0.016 | 0.064 | 0.212 |
| 50 | -2.783 | -1.668 | -1.345 | -0.001 | 0.028 | 0.155 |

Table (6.1). Power estimate for $K^*$ -Statistics.

| Distribution | Parameter θ | Sample size 10 | 20 | 30 |
|---|---|---|---|---|
| F1 (Linear) | 1 | 0.132 | 0.193 | 0.335 |
| | 2 | 0.185 | 0.270 | 0.477 |
| | 3 | 0.217 | 0.328 | 0.572 |
| | 4 | 0.238 | 0.386 | 0.641 |
| F2 (Pareto) | 0.2 | 0.131 | 0.181 | 0.267 |
| | 0.8 | 0.503 | 0.748 | 0.887 |
| | 2 | 0.897 | 0.993 | 1.000 |
| | 3 | 0.965 | 1.000 | 1.000 |
| | 4 | 0.985 | 1.000 | 1.000 |
| F3 (Weibull) | 0.2 | 0.999 | 1.000 | 1.000 |
| | 0.8 | 0.218 | 0.315 | 0.448 |
| | 2 | 0.663 | 0.911 | 0.983 |
| | 3 | 0.934 | 0.984 | 0.997 |
| | 4 | 0.981 | 0.998 | 1.000 |
| F4 (Gamma) | 2 | 0.356 | 0.638 | 0.874 |
| | 3 | 0.676 | 0.946 | 0.996 |
| | 4 | 0.847 | 0.989 | 0.999 |

Table (6.2). Power estimate for $L^*$ -Statistics.

| Distribution | Parameter θ | Sample size 10 | 20 | 30 |
|---|---|---|---|---|
| F1 (Linear) | 1 | 0.507 | 0.737 | 0.811 |
| | 2 | 0.486 | 0.685 | 0.794 |
| | 3 | 0.272 | 0.464 | 0.608 |
| | 4 | 0.164 | 0.299 | 0.429 |
| F2 (Pareto) | 0.2 | 0.360 | 0.481 | 0.615 |
| | 0.8 | 0.705 | 0.881 | 0.956 |
| | 2 | 0.940 | 0.995 | 1.000 |
| | 3 | 0.969 | 0.997 | 1.000 |
| | 4 | 0.985 | 0.998 | 1.000 |
| F3 (Weibull) | 0.2 | 0.965 | 0.996 | 1.000 |
| | 0.8 | 0.374 | 0.495 | 0.620 |
| | 2 | 0.678 | 0.830 | 0.887 |
| | 3 | 0.450 | 0.610 | 0.706 |
| | 4 | 0.267 | 0.396 | 0.522 |
| F4 (Gamma) | 2 | 0.799 | 0.842 | 0.872 |
| | 3 | 0.891 | 0.941 | 0.950 |
| | 4 | 0.954 | 0.972 | 0.972 |

Table (6.3). Power estimate for $M^*$ -Statistics.

| Distribution | Parameter θ | Sample size 10 | 20 | 30 |
|---|---|---|---|---|
| F1 (Linear) | 1 | 0.510 | 0.522 | 0.494 |
| | 2 | 0.470 | 0.497 | 0.501 |
| | 3 | 0.492 | 0.512 | 0.477 |
| | 4 | 0.510 | 0.471 | 0.525 |
| F2 (Pareto) | 0.2 | 0.512 | 0.493 | 0.529 |
| | 0.8 | 0.490 | 0.503 | 0.512 |
| | 2 | 0.500 | 0.497 | 0.472 |
| | 3 | 0.517 | 0.508 | 0.488 |
| | 4 | 0.482 | 0.475 | 0.525 |
| F3 (Weibull) | 0.2 | 0.515 | 0.487 | 0.500 |
| | 0.8 | 0.519 | 0.489 | 0.493 |
| | 2 | 0.507 | 0.510 | 0.493 |
| | 3 | 0.531 | 0.497 | 0.507 |
| | 4 | 0.507 | 0.490 | 0.486 |
| F4 (Gamma) | 2 | 0.473 | 0.499 | 0.489 |
| | 3 | 0.519 | 0.495 | 0.491 |
| | 4 | 0.509 | 0.504 | 0.471 |

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