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$\alpha(x)$-Potentials in an infinite network

Sujith Sivan *

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600 005, India

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Abstract This paper is a unified study of real-valued functions on an infinite network, with results generalizing some of those proved in the case of random walks, finite and infinite electrical networks and Schrödinger operators.
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1. Introduction

Random walks, finite and infinite electrical networks and Schrödinger equations are three of the important topics which make an extensive use of the discrete Laplacian operator and its properties. Though the problems are varied in these three cases, the methods of solving them have generally a common approach based on the properties of the discrete Laplace operator. In this paper, we introduce the notion of $\alpha(x)$-Laplacian operator on the set of real-valued functions defined on an infinite network X, where $\alpha(x)$ denotes some real-valued function on X. The potential theory developed for the $\alpha(x)$-Laplace operator covers the important developments with the above-mentioned three theories.

* Tel.: +91 9962823484.
E-mail address: suji_svn@yahoo.com

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2. Preliminaries

Let X denote an infinite network. That is, X is an infinite graph with a countable number of vertices and a countable number of edges, each edge $e = [x, y]$ joining some pair of vertices x and y; there are no self loops, that is no edge of the form $e = [x, x]$; if $e = [x, y]$ is an edge, then we write $x \sim y$ and say that x and y are neighbours. We assume that X is connected, that is given any two vertices a and b in X there exists a finite path $\{a = a_0, a_1, \ldots, a_n = b\}$ connecting a and b; and that X is locally finite, that is for any vertex x, the number of neighbours of x is finite. For any pair of vertices x and y, there is an associated real number $t(x, y) \geqslant 0$ such that $t(x, y) > 0$ if and only if $x \sim y$. Consequently, note that for any vertex x, $\sum_y t(x, y) = t(x) > 0$. We do not assume that $t(x, y) = t(y, x)$.

Let E be a subset of X. We say that a vertex x in E is an interior vertex of E if and only if x and all its neighbours in X are in E. The set of all interior points of E is denoted by $\overset{\circ}{E}$ and the boundary of E is $\partial E = E \setminus \overset{\circ}{E}$. Let u be a function defined on E. For $x \in \overset{\circ}{E}$, the Laplacian of u at x is defined as

$$\Delta u(x) = \sum_y t(x, y)[u(y) - u(x)].$$

u is said to be Δ-superharmonic (Δ-subharmonic and Δ-harmonic, respectively) on E if and only if $\Delta u(x) \leqslant 0$ ($\Delta u(x) \geqslant 0$ and $\Delta u(x) = 0$, respectively) for every $x \in \overset{\circ}{E}$.

In the case of random walks in a tree T (Cartier, 1972, Anandam and Bajunaid, 2007) which is considered as an infinite network without closed paths with transition probabilities $\{p(x, y)\}$, a real-valued function u is said to be Δ-superharmonic on T if and only if $\sum_y p(x, y)[u(y) - u(x)] \leqslant 0$ for each x. Since $\sum_y p(x, y) = 1$, u is Δ-superharmonic if and only if $u(x) \geqslant \sum_y p(x, y)u(y)$. A function v is said to be Δ-subharmonic if and only if $-v$ is Δ-superharmonic; and a function h is said to be Δ-harmonic if and only if h and $-h$ are Δ-superharmonic.

In the case of an infinite electrical network X, conductance being given by $c(x, y) = c(y, x)$, a real-valued function u is Δ-superharmonic if and only if $\sum_y c(x, y)[u(y) - u(x)] \leqslant 0$ for each $x \in X$. Since $c(x) = \sum_y c(x, y) > 0$ for each x, u is Δ-superharmonic if and only if $u(x) \geqslant \sum_y \frac{c(x,y)}{c(x)} u(y)$.

In the case of the Schrödinger equation $\Delta u(x) = q(x)u(x)$ on X with $q \geqslant 0$, $q \not\equiv 0$, u is said to be q-superharmonic if and only if $\sum_y c(x, y)u(y) \leqslant [c(x) + q(x)]u(x)$ for each x, that is $u(x) \geqslant \sum_y \frac{c(x,y)}{c(x)+q(x)} u(y)$.

If we define $a(x, y) = p(x, y)$ or $\frac{c(x,y)}{c(x)}$ or $\frac{c(x,y)}{c(x)+q(x)}$ depending on the above three cases, then more generally we can deal with these cases in a single study.

3. $\alpha(x)$-Harmonic sheaf

Let X denote an infinite network with a countable number of vertices and a countable number of edges. Denote $x \sim y$ to mean that there is an edge $[x, y]$ joining x

and y, in which case the vertices x and y are said to be neighbours. We assume that X is connected and locally finite. On X, given a system of real numbers $\{a(x, y)\}$ such that $a(x, y) \geqslant 0$ for any pair of vertices x and y and $a(x, y) > 0$ if and only if $x \sim y$. Here $a(x, y)$ may or may not be symmetric. Consider an arbitrary real-valued function $\alpha(x)$ on X. A real-valued function u on a subset E in X is said to be $\alpha(x)$-superharmonic on E if and only if $\alpha(x)u(x) \geqslant \sum_y a(x, y)u(y)$ for every $x \in \overset{\circ}{E}$. Similarly $\alpha(x)$-harmonic functions and $\alpha(x)$-subharmonic functions on E are $\alpha(x)u(x) = \sum_y a(x, y)u(y)$ and $\alpha(x)u(x) \leqslant \sum_y a(x, y)u(y)$ for every $x \in \overset{\circ}{E}$, respectively. If we write $Au(x) = -\alpha(x)u(x) + \sum_y a(x, y)u(y)$, then u is said to be $\alpha(x)$-superharmonic if and only if $Au(x) \leqslant 0$. This system defines a new harmonic sheaf named as $\alpha(x)$-harmonic sheaf. The new harmonic sheaf has lots of properties that are similar to Δ-harmonic sheaf (Abodayeh and Anandam, 2008). Here we prove all the discrete potential-theoritic results related to the $\alpha(x)$-harmonic sheaf in a more general way.

Note 3.1. If $\alpha(x) = 1$ for every $x \in X$, then we have a situation as in random walks. If $\alpha(x) = a(x) = \sum_y a(x, y)$ for every $x \in X$, then we have the situation as in electrical networks. If $\alpha(x) \geqslant a(x)$ for every $x \in X$ and if $\alpha(x_0) > a(x_0)$ for at least one $x_0 \in X$, then we have the situation as in Schrödinger operators.

3.1.1. If s_1 and s_2 are $\alpha(x)$-superharmonic functions on a subset E of X, then $s = \inf\{s_1, s_2\}$ is also $\alpha(x)$-superharmonic on E.

Proof. Let $x_0 \in \overset{\circ}{E}$. Since $s = \inf\{s_1, s_2\}$, assume that $s(x_0) = s_1(x_0)$. Since s_1 is $\alpha(x)$-superharmonic,

$$\alpha(x_0)s(x_0) = \alpha(x_0)s_1(x_0) \geqslant \sum_y a(x_0, y)s_1(y) \geqslant \sum_y a(x_0, y)s(y).$$

Hence s is $\alpha(x)$-superharmonic on E. □

3.1.2. If u_n is a sequence of $\alpha(x)$-superharmonic (or $\alpha(x)$-harmonic) functions on a subset E of X, and if $u(x) = \lim_{n\to\infty} u_n(x)$ exists and is finite for all $x \in E$, then u is $\alpha(x)$-superharmonic (or $\alpha(x)$-harmonic) on E.

Proof. Since u_n is $\alpha(x)$-superharmonic, $Au_n(x) \leqslant 0$. That is,

$$-\alpha(x)u_n(x) + \sum_y a(x, y)u_n(y) \leqslant 0.$$

Taking limits when $n \to \infty$, we obtain

$$-\alpha(x)u(x) + \sum_y a(x, y)u(y) \leqslant 0.$$

Hence u is $\alpha(x)$-superharmonic on E. Similar argument shows that if each u_n is $\alpha(x)$-harmonic on E, then $u = \lim u_n$ is also $\alpha(x)$-harmonic on E. □

3.1.3. If $s \geqslant 0$ is an $\alpha(x)$-superharmonic function on X and $s(x_0) = 0$ for some $x_0 \in X$, then $s \equiv 0$.

Proof. Suppose $s(x_0) = 0$ for some x_0. Since s is $\alpha(x)$-superharmonic, $As(x_0) \leqslant 0$. That is,

$$-\alpha(x_0)s(x_0) + \sum_y a(x_0, y)s(y) \leqslant 0.$$

This implies that $\sum_y a(x_0, y)s(y) \leqslant 0$

andhence $s(y) = 0,$ forevery $y \sim x_0$.

Since X is connected, $s \equiv 0$. □

Note 3.2. If $\alpha(x_0) \leqslant 0$ for some x_0 in X, then there cannot be any positive $\alpha(x)$-superharmonic function on X.

For, suppose there exists a positive $\alpha(x)$-superharmonic function s on X. Then $As(x) \leqslant 0$.

Now,

$$-\alpha(x_0)s(x_0) + \sum_y a(x_0, y)s(y) \leqslant 0.$$

Hence, $\sum_y a(x_0, y)s(y) \leqslant \alpha(x_0)s(x_0) \leqslant 0$

which implies that $s(y) = 0,$ forevery $y \sim x_0$.

This implies that $s \equiv 0$, since X is connected.

Remark 3.3. If there is no positive $\alpha(x)$-superharmonic function on X, then there is no significance in proceeding further with this paper. To prove results in potential theory globally, we need at least one positive $\alpha(x)$-superharmonic function on X. So we assume in the sequel that $\alpha(x) > 0$ for all $x \in X$.

Lemma 3.4. *Let $\mathscr{F} = \{f_i\}_{i\in I}$ be an increasingly filtered family of functions on a network X. Suppose $f(x) = \sup_{i\in I} f_i(x)$ is finite for each $x \in X$. Then there exists an increasing sequence of functions f_n on X, $\{f_n\}$ is a sequence of the family $\mathscr{F}$, such that $f(x) = \lim_{n\to\infty} f_n(x)$.*

Proof. Let $y, z \in X$. Then there exists two increasing sequences of functions $\{f'_n\}$ and $\{f''_n\}$ from $\{f_i\}$ such that $f(y) = \lim_{n\to\infty} f'_n(y)$ and $f(z) = \lim_{n\to\infty} f''_n(z)$. Now $\mathscr{F}$ being a filtered family, choose a subsequence $\{f'''_n\}$ of this family constructed as follows

Let $f_1''' \geqslant \max\{f_1', f_1''\}$.

Inductively define $f_n''' \geqslant \max\{f_n', f_n'', f_{n-1}'''\}$ for $n > 1$. Then $\{f_n'''\}$ is an increasing sequence of functions from the family $\mathscr{F}$. Since $f_n''' \geqslant f_n'$ and $\lim_{n\to y} f_n'(y) = f(y) = \sup_i f_i(y)$, we conclude that $\lim_{n\to\infty} f_n'''(y) = f(y)$. Similarly $\lim_{n\to\infty} f_n'''(z) = f(z)$.

If $t \in X$ is another vertex, then adopting the above procedure we can extract an increasing sequence $\{f_n^{iv}(x)\}$ from the family $\mathscr{F}$ such that $f^{iv} \geqslant f_n'''$ for all n and $f(x) = \lim_{n\to\infty} f^{iv}(x)$ if $x = y$, z or t. Since X is countable, the above procedure can be extended to obtain an increasing sequence $\{f_n\}$ from the family $\mathscr{F}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$. $\square$

Theorem 3.5. *Let $u = \{u_i\}_{i\in I}$ be an increasingly filtered family of $\alpha(x)$-subharmonic ($\alpha(x)$-harmonic, respectively) functions on X such that $u(x) = \sup_i u_i(x)$ is finite for each $x \in X$. Then u is $\alpha(x)$-subharmonic ($\alpha(x)$-harmonic, respectively) on X.*

Proof. By the above Lemma 3.4, there exists an increasing sequence of functions $\{u_n\}$ from u such that $u(x) = \lim_{n\to\infty} u_n(x)$ for every $x \in X$. Since $u_n(x)$ is $\alpha(x)$-subharmonic ($\alpha(x)$-harmonic, respectively) on X, by 3.1.2 u is $\alpha(x)$-subharmonic ($\alpha(x)$-harmonic, respectively) on X. $\square$

Theorem 3.6. *Let $u(x)$ be $\alpha(x)$-superharmonic and $v(x)$ be $\alpha(x)$-subharmonic on a subset E of X such that $v(x) \leqslant u(x)$. Then there exists an $\alpha(x)$-harmonic function $h(x)$ on E such that $v(x) \leqslant h(x) \leqslant u(x)$.*

Proof. Let $\mathfrak{F}$ be the family of all $\alpha(x)$-subharmonic functions s on E such that $s(x) \leqslant u(x)$. Fix $z \in E$. Now

$$s(z) \leqslant \sum_y \frac{a(z, y)}{\alpha(z)} s(y).$$

Take

$$s_z(x) = \begin{cases} s(x) & \text{if } x \neq z \\ \sum_y \frac{a(z,y)}{\alpha(z)} s(y) & \text{if } x = z. \end{cases}$$

Now, $s_z(z) = \sum_y \frac{a(z,y)}{\alpha(z)} s(y) \geqslant s(z)$ and $s_z(x) = s(x)$ if $x \neq z$, which implies $s(x) \leqslant s_z(x)$ for all $x \in X$.

Hence if $y \neq z$, $s_z(y) = s(y) \leqslant \sum_b \frac{a(y,b)}{\alpha(y)} s(b) \leqslant \sum_b \frac{a(y,b)}{\alpha(y)} s_z(b)$ and if $y = z$, $s_z(z) = \sum_b \frac{a(z,b)}{\alpha(z)} s(b) = \sum_b \frac{a(z,b)}{\alpha(z)} s_z(b)$. Hence s_z is $\alpha(x)$-subharmonic on each vertex in E, and $\alpha(x)$-harmonic at the vertex z. Clearly $s_z \leqslant u$ on E. For, $s_z(x) = s(x) \leqslant u(x)$ if $x \neq z$ and if $x = z$, then

$$s_z(z) = \sum_y \frac{a(z,y)}{\alpha(z)} s(y) \leqslant \sum_y \frac{a(z,y)}{\alpha(z)} u(y)$$
$$\leqslant u(z), \text{ since } u \text{ is } \alpha(x) - \text{superharmonic.}$$

Hence $s_z \in \mathfrak{F}$ is an increasingly filtered family of functions each being $\alpha(x)$-harmonic at z and majorized by $u(x)$, we conclude that if $h(x) = \sup_{s \in \mathfrak{F}} s(x)$ then $h(z) = \sup_{y \in \mathfrak{F}} s(z)_\circ = \sup_{z \in \mathfrak{F}} s_z(z)$. Hence h is $\alpha(x)$-harmonic at the vertex z. Since z is arbitrary in E we conclude that h is $\alpha(x)$-harmonic on E and $v \leqslant h \leqslant u$. □

Definition 3.7. Let $s \geqslant 0$ be an $\alpha(x)$-superharmonic function on X. Suppose any $\alpha(x)$-subharmonic function v on X majorized by s is non-positive, then s is said to be an $\alpha(x)$-potential on X.

In other words, a non-negative $\alpha(x)$-superharmonic function p is an $\alpha(x)$-potential if and only if the following condition is satisfied: If v is $\alpha(x)$-subharmonic on X and $v \leqslant p$, then $v \leqslant 0$.

As an immediate consequence of the above Theorem 3.6, we have the following:

Riesz representation: If $s \geqslant 0$ is $\alpha(x)$-superharmonic on X, then s is the unique sum of an $\alpha(x)$-potential p and an $\alpha(x)$-harmonic function $h \geqslant 0$.

Theorem 3.8. *Given a pair of vertices x and y in X there exist two constants $\beta > 0$ and $\gamma > 0$ such that $\beta u(x) \leqslant u(y) \leqslant \gamma u(x)$, for any $\alpha(x)$-superharmonic function $u \geqslant 0$ on X.*

Proof. Let $\{x = x_0, x_1, \ldots, x_n = y\}$ be a path connecting x and y. Since u is $\alpha(x)$-superharmonic,

$$Au(x_0) = -\alpha(x_0)u(x_0) + \sum_z a(x_0, z)u(z) \leqslant 0.$$

Hence,

$$\begin{aligned} \alpha(x_0)u(x_0) &\geqslant \sum_z a(x_0, z)u(z) \\ &\geqslant a(x_0, x_1)u(x_1) \\ \text{and} \quad u(x_0) &\geqslant \tfrac{a(x_0,x_1)}{\alpha(x_0)} u(x_1). \end{aligned}$$

Again,

$$\begin{aligned} \alpha(x_1)u(x_1) &\geqslant \sum_y a(x_1, y)u(y) \\ &\geqslant a(x_1, x_2)u(x_2) \text{ and} \\ u(x_1) &\geqslant \tfrac{a(x_1,x_2)}{\alpha(x_1)} u(x_2). \end{aligned}$$

Therefore,

$$u(x_0) \geqslant \frac{a(x_0, x_1)}{\alpha(x_0)} \frac{a(x_1, x_2)}{\alpha(x_1)} u(x_2).$$

Proceeding further, we arrive at the inequality

$$u(x_0) \geqslant \frac{a(x_0, x_1)}{\alpha(x_0)} \frac{a(x_1, x_2)}{\alpha(x_1)} \cdots \frac{a(x_{n-1}, x_n)}{\alpha(x_{n-1})} u(x_n),$$

which is of the form $u(x) \geqslant \beta u(y)$. The other inequality $u(x) \leqslant \gamma u(y)$ is proved similarly. □

Theorem 3.9. *(Generalized $\alpha(x)$-Dirichlet problem) Let F be an arbitrary set in X. Let $E \subset \mathring{F}$ and f be a real-valued function on $F \setminus E$. Suppose there exists an $\alpha(x)$-superharmonic function $u(x)$ and an $\alpha(x)$-subharmonic function $v(x)$ on f such that $v(x) \leqslant u(x)$ on f, $v(x) \leqslant f(x) \leqslant u(x)$ on $F \setminus E$. Then there exists a function h on f such that $h = f$ on $F \setminus E$, $v \leqslant h \leqslant u$ on f and $Ah(x) = 0$ for every $x \in E$.*

Proof. Let

$$u_0(x) = \begin{cases} f(x) & \text{if } x \in F \setminus E \\ u(x) & \text{if } x \in E. \end{cases}$$

Then u_0 is a function defined on f. Let $z \in E$. Since $E \subset \mathring{F}$ and u is $\alpha(x)$-superharmonic on f,

$$\alpha(z)u_0(z) = \alpha(z)u(z) \geqslant \sum_y a(z, y)u(y) \geqslant \sum_y a(z, y)u_0(y), \text{ since } u_0 \leqslant u \text{ on } F.$$

Hence $Au_0(z) \leqslant 0$ at every vertex in E and $u_0 = f$ on $F \setminus E$. Similarly, if

$$v_0(x) = \begin{cases} f(x) & \text{if } x \in F \setminus E \\ v(x) & \text{if } x \in E, \end{cases}$$

then $Av_0(z) \geqslant 0$ at every vertex $z \in E$ and $v_0 = f$ on $F \setminus E$. Moreover, $v_0 \leqslant u_0$ on f. Let $\mathfrak{F}$ be the family of functions s on f such that $s = f$ on $F \setminus E$, $s \leqslant u$ on f and $As(z) \geqslant 0$ at every vertex $z \in E$. Let $h(x) = \sup_{s \in \mathfrak{F}} s(x)$. Then as in Theorem 3.6 it follows that h is $\alpha(x)$-harmonic at each vertex z in E, $v \leqslant h \leqslant u$ on f so that $h = f$ on $F \setminus E$. □

4. $\alpha(x)$-Potentials

Even if $\alpha(x) > 0$ for every $x \in X$, it is possible that there may not be any positive $\alpha(x)$-superharmonic function on X.

Example 4.1. Let $X = \{0, 1, 2 \ldots\}$, $a(x, y) = 1$ if $x \sim y$ and $\alpha(n) = \frac{1}{2n+1}$ for every $n \geqslant 0$. Let $v \geqslant 0$ be $\alpha(x)$-superharmonic on X. Let $v(0) = b$ and $v(i) = b_i$ for every i. Since $\alpha(0)v(0) \geqslant v(1)$, we have $b \geqslant b_1$.

Now $\alpha(1)v(1) \geqslant v(0) + v(2)$, that is $b_1 \geqslant \frac{1}{3}b_1 \geqslant b + b_2$ which implies $b_1 \geqslant b$. Hence $b_1 = b$. Moreover, $b_1 \geqslant b + b_2$ which implies $b \geqslant b + b_2$; that is $b_2 \leqslant 0$. Since $v \geqslant 0$, $b_2 = 0$. Since $b_2 \geqslant b_1 + b_3$, we have $0 \geqslant b + b_3$ which implies $b = 0$, $b_3 = 0$. Note that for $i \geqslant 1$, if $b_i = 0$, then since $b_{i-1} + b_{i+1} \leqslant b_i$ we should have $b_{i-1} = 0$ and $b_i = 0$. Consequently, $b_i = 0\ \forall i$.

To obtain further significant results on $\alpha(x)$-superharmonic functions on an infinite network X, we need to assume that there exists at least one positive $\alpha(x)$-superharmonic function on X.

Assumption I. There exists a function $\xi > 0$ on X such that

$$\alpha(x)\xi(x) \geqslant \sum_y a(x, y)\xi(y)), \text{ forevery } x \in X.$$

Example 4.2. The above assumption in the case of Schrödinger operators $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$ reads as follows. Recall in this case $\Delta u(x) = \sum_y c(x, y)[u(y) - u(x)] = \left[\sum_y c(x, y)u(y)\right] - c(x)u(x)$ so that $\alpha(x) = c(x) + q(x)$. Hence the assumption takes the form that there exists a function $\xi > 0$ such that

$$[c(x) + q(x)]\xi(x) \geqslant \sum_y c(x, y)\xi(y) \text{ for every } x \in X;$$

that is $q(x) \geqslant \frac{\Delta\xi(x)}{\xi(x)}$ for every $x \in X$. This in particular permits the possibility of q taking non-positive values also.

Theorem 4.3. *(Classical $\alpha(x)$-Dirichlet problem) Let f be an arbitrary set in X and f be a real-valued function on ∂F such that $|f| \leqslant M\xi$ for some $M > 0$. Then there exists a function h on f such that $|h| \leqslant M\xi$ on f, h is $\alpha(x)$-harmonic on f and $h = f$ on ∂F.*

Proof. Take $E = \overset{\circ}{F}$, $v = -M\xi$ and $u = M\xi$ in the Generalized $\alpha(x)$-Dirichlet problem. Then the theorem follows. □

Corollory 4.4. *Let f be an arbitrary finite set of X. Let f be a real-valued function on ∂F. Then there exists an $\alpha(x)$-harmonic function h on f such that $h = f$ on ∂F.*

Proof. Since ∂F is finite we can always find some $M > 0$ such that $|f| \leqslant M\xi$ on ∂F. Then the corollary is a consequence of the Theorem 4.3. □

Theorem 4.5. *Let $f \geqslant 0$ be an arbitrary function on X. Let $\mathscr{F}$ be the family of $\alpha(x)$-superharmonic functions on X such that $s \geqslant f$ on X. If $\mathscr{F}$ is a non-empty family, then*

$Rf(x) = \inf_{s\in\mathscr{F}} s(x)$ is a non-negative $\alpha(x)$-superharmonic function on X such that $Rf(x)$ is an $\alpha(x)$-harmonic function at each vertex in $E = \{z : \ Af(z) \geqslant 0\}$, in particular at each vertex $x \in X$, where $f(x) = 0$.

Proof. Since $\mathscr{F}$ is a decreasingly filtered family of non-negative $\alpha(x)$-superharmonic functions, we prove as in Lemma 3.4 that $Rf(x) = \inf_{s\in\mathscr{F}} s(x)$ is a non-negative $\alpha(x)$-superharmonic function on X. By the Minimum Principle $Rf(x) > 0$ for each x, if $f \not\equiv 0$ on X. Let $z \in E$. Then $Rf(z) \geqslant f(z)$. Let

$$u(x) = \begin{cases} Rf(x) & \text{if } x \neq z \\ \sum\limits_{y} \frac{a(z,y)}{\alpha(z)} Rf(y) & \text{if } x = z. \end{cases}$$

Note $u(x)$ is an $\alpha(x)$-superharmonic function and u is $\alpha(x)$-harmonic at $x = z$ and $u \leqslant Rf$ on X. If $x \neq z$, $u(x) = Rf(x) \geqslant f(x)$; at $x = z$, $u(z) = \sum_y \frac{a(z,y)}{\alpha(z)} Rf(y) \geqslant \sum_y \frac{a(z,y)}{\alpha(z)} f(y) \geqslant f(z)$ since $z \in E$. Hence $u(x) \geqslant f(x)$ for all $x \in X$, that is $u \in \mathscr{F}$ and hence $u \geqslant Rf$ on X. Consequently $u = Rf$ on X. In particular, $Rf(z) = u(z) = \sum_y \frac{a(z,y)}{\alpha(z)} Rf(y)$. That is, $Rf(x)$ is $\alpha(x)$-harmonic at $x = z$. □

Now, if $f(x) = 0$ at some vertex $x \in X$, then $Af(x) = -\alpha(x)f(x) + \sum_y a(x,y)f(y) = \sum_y a(x,y)f(y) \geqslant 0$ and hence $x \in E$. Consequently $Rf(x)$ is $\alpha(x)$-harmonic at each vertex $x \in X$ where $f(x) = 0$.

Theorem 4.6. *Suppose there exists a positive $\alpha(x)$-potential on X. Then given any y, there exists a unique $\alpha(x)$-potential $G_y(x)$ with point $\alpha(x)$-harmonic support $\{y\}$ (that is, $G_y(x)$ is harmonic outside $\{y\}$) such that $AG_y(x) = -\delta_y(x)$ for every $x \in X$.*

Proof. Define f on X such that

$$f(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Then take Rf (as in Theorem 4.5) which is a positive $\alpha(x)$-superharmonic function on X, Rf is $\alpha(x)$-harmonic outside $\{y\}$. Let $A(Rf(y)) = -\beta$ where $\beta > 0$. Define $G_y(x) = \frac{1}{\beta} Rf(x)$. Then $A[G_y(x)] = -\delta_y(x)$ for every $x \in X$.

Note that $Rf(x)$ (and hence $G_y(x)$) is an $\alpha(x)$-potential on X. For by hypothesis, there exists a positive $\alpha(x)$-potential p on X and hence $Rf(x) \leqslant \frac{p(x)}{p(y)}$. Since $Rf(x)$ is dominated by the $\alpha(x)$-potential $\frac{1}{p(y)} p(x)$, it follows that $Rf(x)$ itself is an $\alpha(x)$-potential on X.

Theorem 4.7. *(Domination principle) Let s be a non-negative $\alpha(x)$-superharmonic function on X and p be an $\alpha(x)$-potential on X with $\alpha(x)$-harmonic support on B (that is, $Ap(x) = 0$ for $x \in X \setminus B$). Suppose $s \geqslant p$ on B. Then $s \geqslant p$ on X.*

Proof. Let $q = \inf(s, p)$. Then q is an $\alpha(x)$-potential on X such that $q = p$ on B. Let $u = p - q$ on X. On $X \setminus B$, $Ap(x) = 0$ while $Aq(x) \leqslant 0$ so that $Au(x) \geqslant 0$ on $X \setminus B$. Moreover, $u = 0$ on B and $u \geqslant 0$ on $X \setminus B$ and so that if $x \in B$, then

$$Au(x) = -\alpha u(x) + \sum_y a(x, y)u(y) = 0 + \sum_y a(x, y)u(y) \geqslant 0.$$

Hence $Au(x) \geqslant 0$ on X. That is $u = p - q$ is $\alpha(x)$-subharmonic on X and since $u \leqslant p$, we conclude that $u \leqslant 0$. That is $p \leqslant q$ on X. Consequently $p = q$ on X, so that $s \geqslant p$ on X. □

Corollary 4.8. *If $G_y(x)$ is the $\alpha(x)$-Green's function on X with point harmonic support $\{y\}$, then $G_y(x) \leqslant \frac{G_y(y)}{\xi(y)}\xi(x)$ for every $x \in X$.*

Proof. Since $G_y(x)$ is an $\alpha(x)$-potential with harmonic support $\{y\}$ and $s(x) = \frac{G_y(y)}{\xi(y)}\xi(x)$ is an $\alpha(x)$-superharmonic function majorizing $G_y(x)$ on its harmonic support $\{y\}$, we have by Domination principle, $G_y(x) \leqslant \frac{G_y(y)}{\xi(y)}\xi(x)$ for all $x \in X$. □

5. $\alpha(x)$-Parahyperbolic networks

Assumption I above implies that $\xi > 0$ is an $\alpha(x)$-superharmonic function. Let us denote by $\mathscr{H}$ the $\alpha(x)$-harmonic sheaf on X. Let us define a new harmonic sheaf $\mathscr{H}_\xi$ on X as follows: $u \in \mathscr{H}_\xi$ if and only if $\xi u \in \mathscr{H}$. Let u be $\mathscr{H}_\xi$-harmonic, then ξu is $\mathscr{H}$-harmonic. Hence

$$\alpha(x)\xi(x)u(x) = \sum_y a(x, y)\xi(y)u(y). \quad \text{This can be written as}$$

$$\alpha_\xi(x)u(x) = \sum_y a_\xi(x, y)u(y),$$

where $\alpha_\xi(x) = \alpha(x)\xi(x)$ and $a_\xi(x, y) = a(x, y)\xi(y)$.

Correspondingly the Laplacian operator A_ξ associated with the $\mathscr{H}_\xi$-harmonic sheaf can be written as

$$A_\xi u(x) = -\alpha_\xi(x)u(x) + \sum_y a_\xi(x, y)u(y).$$

Let now $s(x)$ be A_ξ-superharmonic. That is,

$$\alpha_\xi(x)s(x) \geqslant \sum_y a_\xi(x, y)s(y).$$

$$\text{Hence} \quad \alpha(x)[\xi(x)s(x)] \geqslant \sum_y a(x, y)[\xi(y)s(y)].$$

This means that $\xi(x)s(x)$ is A-superharmonic and conversely. That is $s(x)$ is A_ξ-superharmonic if and only if $\xi(x)s(x)$ is A-superharmonic. Thus changes from $\mathscr{H}$-harmonic sheaf to $\mathscr{H}_\xi$-harmonic sheaf and vice versa are facile. However, if the above-mentioned Assumption I is made in the $\mathscr{H}$-harmonic sheaf, then the constant 1 becomes superharmonic in $\mathscr{H}_\xi$-harmonic sheaf. Keeping the facile interchange between $\mathscr{H}$ and $\mathscr{H}_\xi$ sheaves in mind, we simplify Assumption I to read as Assumption II given below.

Assumption II. $\alpha(x) \geqslant \sum_y a(x, y) = a(x)$ for every $x \in X$. That is, the constant 1 is $\alpha(x)$-superharmonic on X.

Now, if 1 is $\alpha(x)$-superharmonic, then by Theorem 3.6 there exists an $\alpha(x)$-harmonic function h on X such that $0 \leqslant h \leqslant 1$. It opens up three possibilities:

1. The constant 1 is harmonic on X. That is $\alpha(x) = a(x)$ for all $x \in X$. This case has been extensively studied in Yamasaki (1979), Soardi (1994), Abodayeh and Anandam (2008). Hence we leave out this case in this paper.
2. The constant 1 is superharmonic but not harmonic on X. In this case, there may be some harmonic function h on X such that $0 < h(x) < 1$ on X. Then as indicated in the beginning of this section, if we consider the associated harmonic sheaf $\mathscr{H}_h$ (that is u is $\mathscr{H}_h$-harmonic if and only if $h(x)u(x)$ is $\mathscr{H}$-harmonic), then 1 is $\mathscr{H}_h$-harmonic on X. We can then carry out the potential-theoretic study on X as in the case of (1).
3. The last case is when 1 is superharmonic but not harmonic on X and there is no harmonic function h on X such that $0 < h(x) < 1$. That is 1 is an $\alpha(x)$-potential on X. This is the case of the $\alpha(x)$-parahyperbolic network which is considered now.

Definition 5.1. (Sujith and Madhu, 2011) We say that X is $\alpha(x)$-parahyperbolic if and only if 1 is an $\alpha(x)$-potential on X.

The importance of the following Minimum Principle in an $\alpha(x)$-parahyperbolic network comes from the fact that it is valid on an arbitrary subset E. The usual Minimum Principle on a general network X is valid only for finite subsets.

Theorem 5.2. *(Minimum Principle) Let E be an arbitrary subset of an $\alpha(x)$-parahyperbolic network X, $E \neq X$. Let u be a lower bounded $\alpha(x)$-superharmonic function on E. If $u \geqslant 0$ on ∂E, then $u \geqslant 0$ on E.*

Proof. Let $v = \inf(u, 0)$ on E. Extend v by 0 outside E. Denote this extended function also by v. Note $v \leqslant 0$ on E and $v = 0$ on ∂E. Hence v is $\alpha(x)$-superharmonic on X and $v \leqslant 0$ on X. Since by assumption v is lower bounded on E, say $u(x) \geqslant -m$, $m > 0$, for every $x \in E$, we have $0 \geqslant v \geqslant -m$ on X. In particular $-v \leqslant m$ on X. Now $-v$ is $\alpha(x)$-subharmonic on X and m is $\alpha(x)$-potential on the $\alpha(x)$-parahyperbolic network X. Hence $-v \leqslant 0$ on X which implies $0 \leqslant v = \inf(u, 0) \leqslant 0$ on E. Hence $u \geqslant 0$ on E. $\square$

Corollary 5.3. *Let E be a proper subset of an $\alpha(x)$-parahyperbolic network. Let h be a bounded $\alpha(x)$-harmonic function on E such that $h = 0$ on ∂E. Then $h = 0$ on E.*

Corollary 5.4. *Let X be an $\alpha(x)$-parahyperbolic network. E is a proper subset of X and f is a bounded real-valued function on ∂E. Then the Dirichlet solution on E with boundary values f on ∂E is unique.*

Proof. Existence: There exists a bounded harmonic function h on E such that $h = f$ on ∂E (Theorem 3.9). Uniqueness: This follows from Corollary 5.3. □

6. Comparison of $\alpha(x)$-superharmonic functions

Comparing the Laplacian operator Δ and the Schrödinger operator Δ_q in an infinite network, we have already noted that the Laplace operator Δ corresponds to the case $\alpha_1(x) = a(x)$ for every $x \in X$ (that is a function u is Δ-superharmonic if and only if u is $a(x)$-superharmonic) and the Schrödinger operator Δ_q corresponds to the case $\alpha_2(x) = a(x) + q(x)$ for every $x \in X$. The relations between Δ and Δ_q can be studied by analysing the $\alpha(x)$ values of the operators, noting that $\alpha_2(x) \geqslant \alpha_1(x)$ for all $x \in X$ and $\alpha_2(x_0) > \alpha_1(x_0)$ for some x_0 in X.

In the sequel, we take $\alpha_1(x)$ and $\alpha_2(x)$ as two real-valued functions on X such that $\alpha_2(x) \geqslant \alpha_1(x)$ for every $x \in X$ and $\alpha_2(x_0) > \alpha_1(x_0)$ for some $x_0 \in X$.

Proposition 6.1. *If $u > 0$ is $\alpha_1(x)$-superharmonic on X, then u is $\alpha_2(x)$-superharmonic on X.*

Proof. Since $u > 0$ is $\alpha_1(x)$-superharmonic, $\alpha_1(x)u(x) \geqslant \sum_y a(x, y)u(y)$. Now $\alpha_2(x)u(x) \geqslant \alpha_1(x)u(x) \geqslant \sum_y a(x, y)u(y)$. Hence u is $\alpha_2(x)$-superharmonic. □

Similarly we prove the following.

Proposition 6.2. *If $u > 0$ is $\alpha_2(x)$-subharmonic, then u is $\alpha_1(x)$-subharmonic.*

Theorem 6.3. *If $u > 0$ is an $\alpha_1(x)$-potential, then u is an $\alpha_2(x)$-potential.*

Proof. By Proposition 6.1, u is an $\alpha_2(x)$-superharmonic function. Let v be an $\alpha_2(x)$-subharmonic function such that $0 \leqslant v \leqslant u$. By Proposition 6.2 any non-negative $\alpha_2(x)$-subharmonic is $\alpha_1(x)$-subharmonic. Hence v is an $\alpha_1(x)$-subharmonic function majorized by an $\alpha_1(x)$-potential u. Hence $v \equiv 0$. Thus u is an $\alpha_2(x)$-potential. □

Theorem 6.4. *Suppose X has positive $\alpha_1(x)$-potentials. Then for any $y \in X$, the $\alpha_1(x)$-Green's potential $G_1(x)$ with point support $\{y\}$ and the $\alpha_2(x)$-Green's potential $G_2(x)$ with the same support $\{y\}$ exists. Also $G_2(x) \leqslant G_1(x)$ for every $x \in X$.*

Proof.

$$A_1G_1(x) = -\alpha_1(x)G_1(x) + \sum_y a(x, y)G_1(y)$$
$$\geqslant -\alpha_2(x)G_1(x) + \sum_y a(x, y)G_1(y), \text{ since } \alpha_2(x) \geqslant \alpha_1(x) = A_2G_1(x).$$

Now, $-\delta_y(x) = A_1G_1(x) \geqslant A_2G_1(x)$. Also $-\delta_y(x) = A_2G_2(x)$. Therefore $A_2G_2(x) \geqslant A_2G_1(x)$, that is $A_2[G_1(x) - G_2(x)] \leqslant 0$ which implies $G_1(x) - G_2(x) = s(x)$ where s is $\alpha_2(x)$-superharmonic. Now $G_1(x) = G_2(x) + s(x)$ which implies $G_2(x) + s(x) \geqslant 0$. That is $-s(x) \leqslant G_2(x)$, hence $-s(x) \leqslant 0$. Consequently $G_1(x) \geqslant G_2(x)$. $\square$

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