ARAB JOURNAL OF MATHEMATICAL SCIENCES

Arab J Math Sci 21 (2015) 170–178

ϕ-semisymmetric generalized Sasakian space-forms

U.C. DE[a], PRADIP MAJHI[b,*]

[a] Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata 700019, West Bengal, India

[b] Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling 734013, West Bengal, India

Received 12 July 2013; received in revised form 18 November 2014; accepted 14 January 2015
Available online 12 February 2015

Abstract. The object of the present paper is to study ϕ-Weyl semisymmetric and ϕ-projectively semisymmetric generalized Sasakian space-forms. Finally, illustrative examples are given.

Keywords: Generalized Sasakian space-form; ϕ-Weyl semisymmetric manifold; ϕ-projectively semisymmetric manifold; Conformally flat; Projectively flat

2010 Mathematics Subject Classification: 53C15; 53C25

1. INTRODUCTION

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine its curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as a real space-form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian space-form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame Alegre, Blair and Carriazo [1] introduced and studied generalized Sasakian space-forms. These space-forms are defined as follows:

* Corresponding author.
E-mail addresses: uc_de@yahoo.com (U.C. De), mpradipmajhi@gmail.com (P. Majhi).
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2015.01.002

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that M is a generalized Sasakian space-form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z = & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{1.1}$$

for any vector fields X, Y, Z on M. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [1] the authors cited several examples of generalized Sasakian space-forms. If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian space-form with Sasakian structure becomes a Sasakian space-form. In [12], Kim studied conformally flat generalized Sasakian space-forms and locally symmetric generalized Sasakian space-forms. He proves some geometric properties of generalized Sasakian space-form which depends on the nature of the functions f_1, f_2 and f_3. A large number of geometers have studied generalized Sasakian space-forms in the papers [2,3,5,4,8]. In [9] De and Sarkar study locally ϕ-symmetric generalized Sasakian space-forms and generalized Sasakian space-forms with η-recurrent Ricci tensor. Also De and Sarkar [10] study projectively flat, projectively semisymmetric generalized Sasakian space-forms. Again in [16] Yildiz and De study ϕ-Weyl semisymmetric and ϕ-projectively semisymmetric non-Sasakian (k, μ)-contact metric manifolds. Motivated by these studies in this paper we study ϕ-Weyl semisymmetric and ϕ-projectively semisymmetric generalized Sasakian space-forms. The present paper is organized as follows:

After preliminaries in Section 3, we consider ϕ-Weyl semisymmetric generalized Sasakian space-forms and obtain necessary and sufficient conditions for a generalized Sasakian space-form to be ϕ-Weyl semisymmetric. Section 4 deals with ϕ-projectively semisymmetric generalized Sasakian space-forms. Finally, illustrative examples are given.

2. Preliminaries

In an almost contact metric manifold we have [6,7]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \phi\xi = 0. \tag{2.1}$$

$$\eta(\xi) = 1, \qquad g(X,\xi) = \eta(X), \qquad \eta(\phi X) = 0. \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(\phi X, X) = 0. \tag{2.4}$$

$$g(\phi X, \xi) = 0. \tag{2.5}$$

Again we know that [1] in a $(2n+1)$-dimensional generalized Sasakian space-form:

$$\begin{aligned} R(X,Y)Z = & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{2.6}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y). \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi. \tag{2.8}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y]. \tag{2.9}$$
$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X]. \tag{2.10}$$
$$S(X,\xi) = 2n(f_1 - f_3)\eta(X). \tag{2.11}$$
$$S(\xi,\xi) = 2n(f_1 - f_3). \tag{2.12}$$
$$Q\xi = 2n(f_1 - f_3)\xi. \tag{2.13}$$
$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.14}$$

where R, S and r denote the curvature tensor of type $(1,3)$, Ricci tensor of type $(0,2)$ and scalar curvature of the space-form respectively.

A $(0,p)$-tensor field T on (M,g) is called parallel when it is invariant under parallel translation, that is, when

$$\nabla T = 0,$$

in particular, if the Riemann–Christoffel curvature tensor R is parallel, that is,

$$\nabla R = 0,$$

then M is said to be locally symmetric.

This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [14]. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature.

A Riemannian manifold (M^{2n+1},g) is said to be semisymmetric if its curvature tensor R satisfies $R(X,Y).R = 0$, X, $Y \in \chi(M)$, where $R(X,Y)$ acts on R as a derivation [13]. Every symmetric space is semisymmetric, but the converse is not true, in general.

For a $(2n+1)$-dimensional Riemannian manifold the Weyl conformal curvature tensor is defined by [15]

$$\begin{aligned} C(X,Y)Z = {} & R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)QX - g(X,Z)QY\} \\ & + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{2.15}$$

where r is a scalar curvature and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

After the conformal curvature tensor, the projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of a Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. The projective curvature tensor is defined by [15]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}, \tag{2.16}$$

where S is the Ricci tensor of M.

Definition 2.1. An almost contact metric manifold $(M^{(2n+1)}, g)$, $n > 1$, is said to be ϕ-Weyl semisymmetric if

$$C(X, Y).\phi = 0$$

on M.

Definition 2.2. An almost contact metric manifold $(M^{(2n+1)}, g)$, $n > 1$, is said to be ϕ-projectively semisymmetric if

$$P(X, Y).\phi = 0$$

on M, for all $X, Y \in \chi(M)$.

3. ϕ-Weyl semisymmetric generalized Sasakian space-forms

Let M be a $(2n+1)$-dimensional $(n > 1)$ ϕ-Weyl semisymmetric generalized Sasakian space-forms. Therefore $C(X, Y).\phi = 0$ turns into

$$(C(X, Y).\phi)Z = C(X, Y)\phi Z - \phi C(X, Y)Z = 0, \tag{3.1}$$

for any vector fields X, Y and $Z \in \chi(M)$.

Now, from (2.15) it follows that

$$\begin{aligned} C(X, Y)\phi Z = {} & R(X, Y)\phi Z - \frac{1}{2n-1}\{S(Y, \phi Z)X - S(X, \phi Z)Y \\ & + g(Y, \phi Z)QX - g(X, \phi Z)QY\} \\ & + \frac{r}{2n(2n-1)}\{g(Y, \phi Z)X - g(X, \phi Z)Y\}. \end{aligned} \tag{3.2}$$

Using (2.6)–(2.8) in (3.2) yields

$$\begin{aligned} C(X, Y)\phi Z = {} & f_1\{g(Y, \phi Z)X - g(X, \phi Z)Y\} \\ & + f_2\{g(X, \phi^2 Z)\phi Y - g(Y, \phi^2 Z)\phi X + 2g(X, \phi Y)\phi^2 Z\} \\ & + f_3\{\eta(X)\eta(\phi Z)Y - \eta(Y)\eta(\phi Z)X \\ & + g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi\} \\ & - \frac{(2n-1)f_1 + 3f_2}{(2n-1)}\{g(Y, \phi Z)X - g(X, \phi Z)Y\} \\ & + \frac{3f_2 + (2n-1)f_3}{(2n-1)}\{g(Y, \phi Z)\eta(X)\xi - g(X, \phi Z)\eta(Y)\xi\}. \end{aligned} \tag{3.3}$$

By virtue of (2.1) we have from (3.3)

$$\begin{aligned} C(X, Y)\phi Z = {} & -\frac{3f_2}{(2n-1)}\{g(Y, \phi Z)X - g(X, \phi Z)Y\} \\ & + f_2\{-g(X, Z)\phi Y + \eta(Z)\eta(X)\phi Y + g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X \\ & - 2g(X, \phi Y)Z + 2g(X, \phi Y)\eta(Z)\xi\} \\ & - \frac{3f_2}{(2n-1)}\{g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi\}. \end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned}\phi C(X,Y)Z = &-\frac{3f_2}{(2n-1)}\{g(Y,Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y \\ &- \eta(Y)\eta(Z)\phi X\} + f_2\{g(Y,\phi Z)X - g(X,\phi Z)Y \\ &+ g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi \\ &- 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi\}.\end{aligned} \tag{3.5}$$

Substituting (3.4), (3.5) in (3.1) yields

$$\begin{aligned}&\frac{2(n+1)}{(2n-1)}f_2\{-g(Y,\phi Z)X + g(X,\phi Z)Y - g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)\eta(X)\xi \\ &\quad + g(Y,Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\} = 0,\end{aligned} \tag{3.6}$$

which implies either $f_2 = 0$ or,

$$\begin{aligned}&-g(Y,\phi Z)X + g(X,\phi Z)Y - g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)\eta(X)\xi \\ &\quad + g(Y,Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X = 0.\end{aligned} \tag{3.7}$$

Taking the inner product by W of (3.7) we obtain

$$\begin{aligned}&-g(Y,\phi Z)g(X,W) + g(X,\phi Z)g(Y,W) \\ &\quad - g(X,\phi Z)\eta(Y)\eta(W) + g(Y,\phi Z)\eta(X)\eta(W) \\ &\quad + g(Y,Z)g(\phi X,W) - g(X,Z)g(\phi Y,W) + \eta(X)\eta(Z)g(\phi Y,W) \\ &\quad - \eta(Y)\eta(Z)g(\phi X,W) = 0.\end{aligned} \tag{3.8}$$

Putting $Y = W = e_i$, where $\{e_i, \xi\}$, $(1 \le i \le 2n)$ is the orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$(2n-3)g(X,\phi Z) = 0, \tag{3.9}$$

which implies that

$$g(X,\phi Z) = 0, \tag{3.10}$$

which is a contradiction. Conversely, if $f_2 = 0$ then from (3.6) it follows that the space-form is ϕ-Weyl semisymmetric. Therefore from the above discussion we can state the following:

Theorem 3.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space-form $M(f_1, f_2, f_3)$ is ϕ-Weyl semisymmetric if and only if $f_2 = 0$.*

In [12] U.K. Kim proved that for a $(2n+1)$-dimensional generalized Sasakian space-form the following holds:

(i) If $n > 1$, then M is conformally flat if and only if $f_2 = 0$.

(ii) If M is conformally flat and ξ is a Killing vector field, then M is locally symmetric and has constant ϕ-sectional curvature.

In view of the first part of the above theorem of Kim we obtain the following:

Corollary 3.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space-form $M(f_1, f_2, f_3)$ is ϕ-Weyl semisymmetric if and only if it is conformally flat.*

Also, in view of the second part of the above theorem of Kim we immediately get the following:

Corollary 3.2. *For a $(2n + 1)$-dimensional $(n > 1)$ ϕ-Weyl semisymmetric generalized Sasakian space-form $M(f_1, f_2, f_3)$ with ξ as a Killing vector field is locally symmetric and has constant ϕ-sectional curvature.*

Hence from Theorem 3.1 and Corollary 3.1 we can state the following:

Corollary 3.3. *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian space-form. Then the following statements are equivalent:*
(i) *M is ϕ-Weyl semisymmetric;*
(ii) *M is conformally flat;*
(iii) $f_2 = 0$.

Recently, in [1] P. Alegre, D. Blair and A. Carriazo prove if a generalized Sasakian space-form $M(f_1, f_2, f_3)$ is a Sasakian manifold, then the functions f_1, f_2, f_3 are constant and $f_1 - 1 = f_2 = f_3$.

Now, in this case $f_2 = 0$ implies $f_3 = 0$ and $f_1 = 1$. Thus from (1.1) we obtain $R(X, Y)Z = g(Y, Z)X - g(X, Z)Y$, that is, the manifold is of constant curvature 1. It is known that if a $(2n + 1)$-dimensional $(n > 1)$ Riemannian manifold is of constant curvature, then the manifold is conformally flat. Also conformally flatness implies ϕ-Weyl semisymmetric. Hence we can state the following:

Corollary 3.4. *A $(2n+1)$-dimensional $(n > 1)$ Sasakian manifold is ϕ-Weyl semisymmetric if and only if the manifold is of constant curvature 1.*

4. ϕ-PROJECTIVELY SEMISYMMETRIC GENERALIZED SASAKIAN SPACE-FORMS

Let M be a $(2n + 1)$-dimensional $(n > 1)$ ϕ-projectively semisymmetric generalized Sasakian space-forms. Therefore $P(X, Y).\phi = 0$ turns into

$$(P(X, Y).\phi)Z = P(X, Y)\phi Z - \phi P(X, Y)Z = 0, \tag{4.1}$$

for any vector fields X, Y and $Z \in \chi(M)$.

Now,

$$P(X, Y)\phi Z = R(X, Y)\phi Z - \frac{1}{2n}\{S(Y, \phi Z)X - S(X, \phi Z)Y\}. \tag{4.2}$$

Using (2.7) in (4.2) we obtain

$$\begin{aligned} P(X, Y)\phi Z = {} & R(X, Y)\phi Z - \frac{1}{2n}[(2nf_1 + 3f_2 - f_3)\{g(Y, \phi Z)X \\ & - g(X, \phi Z)Y\}]. \end{aligned} \tag{4.3}$$

By virtue of (2.6) we obtain from (4.3)

$$\begin{aligned}P(X,Y)\phi Z = &-\frac{(3f_2-f_3)}{2n}\{g(Y,\phi Z)X - g(X,\phi Z)Y\}\\ &+ f_2\{g(Y,Z)\phi X - g(X,Z)\phi Y\\ &+\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\\ &- 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi\}\\ &+ f_3\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\}.\end{aligned} \tag{4.4}$$

Similarly,

$$\begin{aligned}\phi P(X,Y)Z = &-\frac{(3f_2-f_3)}{2n}\{g(Y,Z)\phi X - g(X,Z)\phi Y\}\\ &-\frac{(3f_2-f_3)}{2n}\{\eta(X)\eta(Z)\phi Y - \eta(Z)\eta(Z)\phi X\}\\ &+ f_2\{g(Y,\phi Z)X - g(X,\phi Z)Y\}\\ &+ f_2\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\}\\ &+ f_2\{-2g(X,\phi Y)Z = g(X,\phi Y)\eta(Z)\xi\}.\end{aligned} \tag{4.5}$$

Substituting (4.4), (4.5) in (4.1) yields

$$\begin{aligned}&\frac{(2n+3)f_2-f_3}{2n}\{-g(Y,\phi Z)X + g(X,\phi Z)Y + g(Y,Z)\phi X - g(X,Z)\phi Y\\ &\quad+\eta(Z)\eta(X)\phi Y - \eta(Y)\eta(Z)\phi X\}\\ &\quad+(f_3-f_2)\{g(X,\phi Z)\eta(y)\xi - g(Y,\phi Z)\eta(X)\xi\} = 0.\end{aligned} \tag{4.6}$$

Putting $Y=\xi$ in (4.6) we obtain

$$\{(2n-1)f_3 + 3f_2\}g(X,\phi Z)\xi = 0, \tag{4.7}$$

which implies that $f_3 = \frac{3f_2}{1-2n}$. Hence we can state the following:

Theorem 4.1. *For a $(2n+1)$-dimensional ϕ-projectively semisymmetric $(n>1)$ generalized Sasakian space-form $M(f_1, f_2, f_3)$, $f_3 = \frac{3f_2}{1-2n}$ holds.*

In a recent paper [10] De and Sarkar proved the following:

Theorem 4.2 (*[10]*)**.** *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian space-form $M(f_1, f_2, f_3)$ is projectively flat if and only if $f_3 = \frac{3f_2}{1-2n}$.*

Suppose $f_3 = \frac{3f_2}{1-2n}$ holds. Therefore $P=0$ and hence $P(X,Y).\phi = 0$. Thus in view of Theorem 4.1 we can state the following:

Theorem 4.3. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian space-form $M(f_1, f_2, f_3)$ is ϕ-projectively semisymmetric if and only if $f_3 = \frac{3f_2}{1-2n}$.*

From Theorem 7.2 of [11], we note that a $(2n+1)$-dimensional $(n>1)$ generalized Sasakian space-form is Ricci semisymmetric if and only if $f_3 = \frac{3f_2}{1-2n}$. In virtue of Theorem 4.3, we immediately state the following:

Corollary 4.1. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian space-form $M(f_1, f_2, f_3)$ is ϕ-projectively semisymmetric if and only if it is Ricci semisymmetric.*

Also from Theorems 4.2, 4.3 and Corollary 4.1 we can state the following:

Corollary 4.2. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional $(n>1)$ generalized Sasakian space-form. Then the following statements are equivalent:*
(i) *M is ϕ-projectively semisymmetric;*
(ii) *M is projectively flat;*
(iii) *M is Ricci semisymmetric;*
(iv) $f_3 = \frac{3f_2}{1-2n}$.

5. Examples

Examples 1. In [1], it was shown that the warped product $\mathbb{R} \times_f \mathbb{C}^m$ is a generalized Sasakian space-form with

$$f_1 = -\frac{(f')^2}{f^2}, \qquad f_2 = 0, \qquad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$, $t \in \mathbb{R}$ and f' denotes the derivative of f with respect to t. If we choose $m = 4$ and $f(t) = e^t$, then $M(f_1, f_2, f_3)$ is a 5-dimensional conformally flat generalized Sasakian space-form, since $f_2 = 0$. Therefore all the equivalent conditions of Corollary 3.3 are verified.

Examples 2. Let $N(a, b)$ be a generalized complex space-form of dimension 4, then by the warped product $M = \mathbb{R} \times N$ endowed with the almost contact metric structure (ϕ, ξ, η, g_f) is a generalized Sasakian space-form $M(f_1, f_2, f_3)$ [1] with

$$f_1 = \frac{a-(f')^2}{f^2}, \qquad f_2 = \frac{b}{f^2}, \qquad f_3 = \frac{a-(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$, $t \in \mathbb{R}$ and f' denotes the derivative of f with respect to t. If we choose $a = 0$, $b = 1$ and $f(t) = t$ with $t > 0$, then $f_1 = -\frac{1}{t^2}$, $f_2 = \frac{1}{t^2}$ and $f_3 = -\frac{1}{t^2}$. Hence $f_2 = -f_3$.

On the other hand, from Corollary 4.2 for $(2n+1)$-dimension generalized Sasakian space-form $M(f_1, f_2, f_3)$ we have $f_3 = \frac{3f_2}{1-2n}$. Therefore in dimension 5, that is, for $n = 2$ we have $f_2 = -f_3$. So all the equivalent conditions of Corollary 4.2 are verified.

Acknowledgements

The authors are thankful to the referees for their comments and valuable suggestions towards the improvement of this paper.

References

[1] P. Alegre, D.E. Blair, A. Carriazo, Generalized Sasakian space-forms, Israel J. Math. 141 (2004) 157–183.
[2] P. Alegre, A. Carriazo, Submanifolds of generalized Sasakian space-forms, Taiwanese J. Math. 13 (2009) 923–941.

[3] P. Alegre, A. Carriazo, Generalized Sasakian space-forms and conformal changes of metric, Results Math. 59 (2011) 485–493.
[4] P. Alegre, A. Carriazo, Structures on generalized Sasakian space-forms, Differential Geom. Appl. 26 (2008) 656–666.
[5] P. Alegre, A. Carriazo, C. Özgür, S. Sular, New examples of generalized Sasakian space-forms, Proc. Est. Acad. Sci. 60 (2011) 251–257.
[6] D.E. Blair, Lecture Notes in Mathematics, vol. 509, Springer-Verlag, Berlin, 1976.
[7] D.E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Birkhauser, Boston, 2000.
[8] U.C. De, P. Majhi, Certain curvature properties of generalized Sasakian space-forms, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. 83 (2013) 137–141.
[9] U.C. De, A. Sarkar, Some results on generalized Sasakian space-forms, Thai J. Math. 8 (2010) 1–10.
[10] U.C. De, A. Sarkar, On the projective curvature tensor of generalized Sasakian space-forms, Quaest. Math. 33 (2010) 245–252.
[11] U.C. De, A. Sarkar, Some curvature properties of generalized Sasakian space-forms, Lobachevskii J. Math. 33 (2012) 22–27.
[12] U.K. Kim, Conformally flat generalized Sasakian space-forms and locally symmetric generalized Sasakian space-forms, Note Mat. 26 (2006) 55–67.
[13] O. Kowalski, An explicit classification of 3-dimensional Riemannian spaces satisfying $R(X,Y).R = 0$, Czechoslovak Math. J. 46 (1996) 427–474.
[14] B. O'Neill, Semi-Riemannian geometry. With applications to relativity, in: Pure and Applied Mathematics, vol. 103, Academic Press, Inc., New York, 1983.
[15] K. Yano, M. Kon, Structures on manifolds, in: Series in Pure Math, vol. 3, World Scientific Publ. Co., Singapore, 1984.
[16] A. Yildiz, U.C. De, A classification of (k,μ)-contact metric manifolds, Commun. Korean Math. Soc. 27 (2012) 327–339.